A

TREATISE

ON THE

HIGHER PLANE CURVES:

INTENDED AS A SEQUEL

TO

A TREATISE ON CONIC SECTIONS.

BY THE

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DUBLIN:

HODGES AND SMITH, GRAFTON-STREET,

BOOKSELLERS TO THE UNIVERSITY.

MDCCCLII.

DUBLIN:
Printed at the University Press,
BY M. H. GILL.

PREFACE.

THE present volume forms the completion of a Treatise on the Analytic Geometry of Space of Two Dimensions, of which the first Part (containing the theory of Conic Sections) was published about four years ago. It was my wish that the work should be elementary enough to serve as a book of instruction to students commencing without any previous knowledge of the science, and at the same time should contain a sufficiently complete account of what is known on the subject of Curves to be capable of serving as a book of reference to more advanced readers. Considered merely in the former light, I am aware that the book is too long, and that many who take it up will be startled at the sight of so many pages devoted to curves of higher dimensions. Now, though I have inserted nothing but what I have thought might be acceptable to some of those readers who take peculiar interest in the subject of Geometry, I am far from desiring that students in general should spend, in the perusal of the entire of this Treatise, time which ought to be devoted to more important branches of Mathematics. But I am of opinion that ordinary students, without spending more time on Analytic Geometry than they commonly do at present, might distribute that time more profitably. When once a student has worked through

examples enough to make him expert in the practice of the method of co-ordinates, but little benefit is derived from the multiplication of isolated theorems; and he may with advantage bestow on the study of general methods, and of the general properties of Curves, some of that time which is now commonly given up to endless properties of Conic Sections. Such readers may confine their attention to the second Chapter of this volume: if in other studies they should require a knowledge of the properties of any particular curve, they will be able to find, by the help of the Index, the account which I have given of it.

Considered as a book for advanced readers, I regret that I have not had leisure for the reading necessary to make this work as complete as it ought to be. My knowledge of the older writers on Geometry is, for the most part, either second-hand or superficial; and I cannot even claim acquaintance with all of the most remarkable works and memoirs which of late years have been published on the subject. But even such an imperfect compilation as I have been able to make will, I hope, not be useless. In every branch of Mathematics it is desirable that elementary treatises should keep pace with the progress of science. A new province can scarcely be considered as gained to analysis until roads have been made through the conquered territory. It is desirable that each new student who wishes to devote himself to original investigation in any branch of Mathematics should have his energies as quickly as possible brought to bear upon the undiscovered parts of the science; and that his abilities should not be wasted in rediscovering truths known already, nor his time frittered away in hunting for them without a guide through the wilderness of scientific periodicals.

The plan upon which I have proceeded has been, to take the less recent Geometry as represented in previous elementary treatises (Lardner's Algebraic Geometry, and the Chapters on Curves in Gregory's Examples, being those which I have most frequently consulted), and to incorporate with this what appeared to me most important in the works of modern Geometers. Poncelet's Traité des Propriétés Projectives and Chasles's Aperçu Historique were principally useful in the first Part of this Treatise, but have also afforded some materials for the present volume. I had hoped to have derived much assistance from M. Chasles's long-promised Traité de Géométrie Supérieure, more especially as he is understood to have given much attention to Curves of the third degree; but his work, though daily expected here, has been delayed too long for me to benefit by it. Plücker's works on Analytic Geometry I was not acquainted with when the first edition of the *Conics* was published, but I have repeatedly had occasion to acknowledge my obligations to them in the following pages. And I have, besides, made use of the Geometrical Papers which seemed to me most interesting in the later volumes of Crelle's and Liouville's Journals, and in the Cambridge and Dublin Mathematical Journal.

It has been my general practice to give references where I am indebted to the labours of other writers, but it must not be supposed that I claim as my own those places where no references are given. Some references have been omitted through oversight, many more, in the case of theorems, with which I have been so long familiar that I have forgotten where I met them first; and besides, it has so often occurred to me to suffer from the depredations of those worst of plagiarists, "qui nostra ante

nos dixerunt," that I feel my reading has not been extensive enough to entitle me to claim any theorem for myself, even when I do not remember having seen it published before. Where I have been most indebted to others I think it will be found that I have not copied servilely, and that I have used their labours to guide rather than to supersede my own investigations. There is, at all events, enough of the book my own to make me apprehensive that several errors may be found in it. These I leave to the indulgent correction of such readers as have had experience enough to know how difficult it is to escape such slips. The inaccuracies of the work would have been more numerous than they are, but for the kind assistance of my friend, Dr. Hart, who has not only furnished me with several contributions, acknowledged in their proper places, but who has also taken the trouble of reading the proofs of the entire work, and who has been always ready to oblige me with his advice on any point of which I felt doubtful.

I take this opportunity of repeating my acknowledgment to Mr. Townsend for similar assistance afforded me in the preparation of the former volume on Conic Sections.

The following is an account of the arrangement which I have adopted in this Treatise. Having in the *Conics* sufficiently explained those co-ordinate systems in which the position of a line is expressed by a single equation, and that of a point by two equations, I have in the first Chapter given an account of those methods in which it is the position of a point which is expressed by a single equation. This Chapter, being quite detached from the rest, may be either altogether omitted by the student, or may be reserved until he has read some of the following Chapters. I have thought it advisable, however, to place

it at the commencement, that the student may be aware from the first of the double interpretation which can be given to every equation employed afterwards. The second Chapter is devoted to an account of the general properties of all algebraic curves. I commence with those which may be derived as consequences from the number of terms in the general equation. Were the book only intended for advanced readers, I should then have proceeded at once to the investigation of the theory of multiple points by the method used in Sect. IV. For the sake of learners, however, I have prefixed illustrations of singular points by particular examples, which I think will give him clearer ideas of the different possible species of singular points than are afforded by the methods usually followed in works on the Differential Calculus. For the determination of multiple points and tangents in general, I have used (Art. 61, &c.) the method which was introduced by M. Joachimsthal (Crelle, xxxiv. 34) for obtaining the equation of the tangents to a Curve from a given point, and which seems of great importance in the symmetrical investigation of the general properties of Curves. Having by this method obtained the general theory of the singularities of Curves, I proceed to explain how to find the equation of the envelope of a line whose equation involves a variable parameter, using as examples the problems of finding the reciprocal, the evolute, or the caustics, of a given curve. On the subject of evolutes I have given some considerations, determining their degree and class in general, notwithstanding some doubts expressed by M. Terquem (Journal, ix. p. 275) as to the trustworthiness of my arguments, if unsustained by other proofs. The Chapter concludes with an account of some of the properties of those points which for higher Curves

are analogous to the foci of Conic Sections. The third Chapter is occupied with the special properties of Curves of the third degree. The first section contains those properties which can be derived as easy consequences from some of the principal forms which the general equation may assume. Having in the second section given some of the chief properties of the points of inflexion, I have in the third section, acting on a suggestion of Dr. Hart's, somewhat modified, given an analysis of the different possible figures which a curve of the third degree can assume. Previous to examination I had supposed that the varieties were too numerous to be retained in any memory, but the principle of classification here adopted seems sufficiently simple to be easily understood and remembered. Sections IV. V. and VIII. are developments for Curves of the third degree of theorems generally proved in the preceding Chapter. Sections VI. and VII. contain methods applicable when the curve has a double point. I reserved for the last section, as less inviting to ordinary readers, the discussion of the general equation of the third degree. The fourth and fifth Chapters contain the principal known properties of Curves of the fourth degree, and of the more remarkable transcendental Curves. In the sixth Chapter I give an account of some general methods of obtaining the properties of Curves, and particularly of these methods by which the properties of one curve are deduced from those of another. Throughout this volume the student is supposed to be acquainted with the ordinary processes of the Differential Calculus; but it is only in the last Chapter that I discuss those problems which require the use of the Integral Calculus, such as the rectification and quadrature of curves. I had been doubtful whether I might

not have contented myself with referring the reader for information on this subject to works on the Calculus itself; but it seems the more suitable, and probably will be the ultimate arrangement, that works on the Calculus should contain its general principles and rules, but that its application to particular sciences should be found in special treatises on those sciences. To the ordinary examples given on this subject I have added some taken from the later volumes of Liouville's Journal; and the reader will perceive from the acknowledgments in the Chapter, that the Messrs. Roberts have kindly allowed me to profit by their greater familiarity with this branch of the Theory of Curves.

In bringing out a work addressed to too narrow a circle of readers to be likely to be remunerative, I am under much obligation to the Board of Trinity College for extending to me the liberality with which they have constantly been ready to contribute to the expenses of similar works published at their Press.

The reader is requested to take notice that the references to the *Conic Sections* are made to the second edition.

CONTENTS.

THEORY OF THE HIGHER PLANE CURVES.

CHAPTER I.

TANGENTIAL CO-ORDINATES.

ART. 1. There are two points of view, both equally natural, from which curves may be considered, and according to which they may be classified. A curve may be considered as the locus of a moveable point, and may be classed according to the number of such points which lie on any assumed right line; or a curve may be considered as the envelope of a moveable right line, and may be classed according to the number of such lines (tangents to the curve) which can be drawn through any assumed point. Both these principles of classification have been adopted by modern geometers. A curve is said to be of the n^{th} *degree* or *order* when any right line meets it in n points; and of the n^{th} *class* when n tangents can be drawn to it through any assumed point. A conic section, for instance, is a curve both of the second degree and of the second class; but in the case of curves of higher orders it will ordinarily not happen that the degree and class of the curve will be the same. It is evident (*Conics*, p. 253) that the degree of a curve is the same as the class of its reciprocal, and *vice versâ*.

In the Cartesian and trilinear systems of co-ordinates, which we have hitherto adopted, the degree of the equation at once indicates the degree of the curve, while the class cannot be so directly ascertained; we proceed now to show that there are other systems of co-ordinates in which it is the class of the curve which is indicated by the degree of the equation. In the former systems the

position of a point was determined by certain co-ordinates, while that of a right line was represented by an equation; in the systems which we are now about to explain, the position of a right line is represented by co-ordinates, while that of a point is indicated by an equation.

2. One such system of co-ordinates we might at once form from the ordinary Cartesian or trilinear methods. Let the Cartesian equation of a right line be $Ax + By + C = 0$, then if the quantities A, B, C were known, the position of the right line would be known. We may call these three quantities (which are only equivalent to two independent variables, since we are only concerned with the ratios $A : B$, $A : C$) the co-ordinates of the right line.

It was proved (*Conics*, p. 64) that if the constants in the equation of a right line fulfil any linear relation, $lA + mB + nC = 0$, then the right line will pass through a fixed point. We may call the equation $lA + mB + nC = 0$ the equation of that point.

Lastly, let the constants A, B, C be connected by a relation of the n^{th} degree,

$$aA^n + bA^{n-1} B + \&c. = 0.$$

The line $Ax + By + C = 0$, moving about subject to this condition, will envelope a certain curve, of which the above may be regarded as the equation. This curve will be of the n^{th} class, or such that n tangents can be drawn to it from any given point: for take any point, $lA + mB + nC = 0$, and by the help of this equation eliminate A from the given equation of the n^{th} degree, and there plainly results an equation of the n^{th} degree to determine $B : C$, the co-ordinates of the tangents which can be drawn through the given point.

As an illustration of this method, take the equation

$$xx' + yy' - r^2 = 0,$$

this line we know to be the pole of $x'y'$ with regard to the circle $x^2 + y^2 = r^2$; now suppose that we were given any relation $\phi(x'y') = 0$; according to one method of interpreting this equation, $\phi(x'y') = 0$ is the Cartesian equation of a curve on which $x'y'$ moves: but if, according to the method of the present Article, we look on $x'y'$ merely as coefficients in the equation $xx' + yy' - r^2 = 0$, then

$\phi(x'y') = 0$ is the tangential equation of the curve enveloped by the latter line: a curve the polar reciprocal of the other curve.

Now if it be required to prove any property of the second curve, it is not necessary first to deduce from $\phi(xy) = 0$, considered as a Cartesian equation, the corresponding property of the first curve, and then infer, by reciprocal polars, the required property of the second; but we may infer the latter directly from $\phi(xy) = 0$, considered as a tangential equation. We shall return presently to follow up this remark; but we shall not dwell further on the system of tangential co-ordinates explained in this Article, since it is based on, and requires a previous knowledge of the Cartesian method. We prefer to pass on to another system of co-ordinates, by which it will more plainly appear that it is just as natural to represent a right line by co-ordinates and a point by an equation, as *vice versâ;* and that the only reason why it should appear to us otherwise is our early familiarity with the Cartesian method.

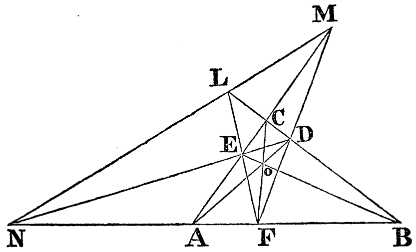
3. In the method of trilinear co-ordinates (*Conics*, p. 225) the position of any point is determined by its distances from three fixed lines. In the system which we are now about to explain, the position of any right line is determined by its distances, α, β, γ, from three fixed points, A, B, C; for it is plain that if the ratios of these were known, the position of the right line would be completely determined.

If the perpendiculars from two points, A, B, on any line, be α, β, it was proved (*Conics*, p. 57) that the perpendicular on it from the point which cuts the line AB in the ratio of $m : l$ will be $\dfrac{l\alpha + m\beta}{l + m}$; consequently for every line which passes through this point the relation will be fulfilled, $l\alpha + m\beta = 0$. We may then consider this as the equation of a point situated on the line joining the points whose equations are $\alpha = 0$, $\beta = 0$, and cutting this line in the ratio $m : l$.

In like manner, if the point $l\alpha + m\beta$ be joined to γ, and cut in the ratio $n : l + m$, the perpendicular on any line from the point so found will be

$$\frac{(l+m)\dfrac{l\alpha + m\beta}{l+m} + n\gamma}{l+m+n} = \frac{l\alpha + m\beta + n\gamma}{l+m+n}.$$

And if $l\alpha + m\beta + n\gamma = 0$, the line fulfilling this condition will pass through the point so found. It follows then that the general equation of the first degree in $\alpha\beta\gamma$ represents a point. The point may be constructed either by cutting BC in the ratio $n:m$, and AD :: $n+m:l$; or by cutting AC :: $l:n$ and BE, :: $l+n:m$; or by cutting AB :: $m:l$, and CF :: $l+m:n$. We are led by this construction to the well-known theorem that the continued product $=1$ of the ratios BD : DC, CE : AE, and AF : FB.



Since the ratio of the areas of the triangles AOB : AOC is the same as that of BD : DC, we may write down the equation of the point O in the form

$$\text{BOC} \cdot \alpha + \text{COA} \cdot \beta + \text{AOB} \cdot \gamma = 0,$$

and we see that the above relation will subsist between the perpendiculars let fall from the points A, B, C on any line which passes through O.

Or, substituting for each triangle BOC its value BO · OC sin BOC, the equation of any point may be written

$$\frac{\sin \text{BOC}}{\text{OA}}\alpha + \frac{\sin \text{COA}}{\text{OB}}\beta + \frac{\sin \text{AOB}}{\text{OC}}\gamma = 0.$$

4. We go on to give some illustrations of the practical application of this method of co-ordinates. Let $u=0$, $v=0$, be the equations of any two points; then any equation of the form $u+kv=0$ denotes a point on the line joining the first two. For if $u = l\alpha + m\beta + n\gamma$, $v = l_1\alpha + m_1\beta + n_1\gamma$, then

$$\text{A}\,\frac{l\alpha + m\beta + n\gamma}{l+m+n} + \text{B}\,\frac{l_1\alpha + m_1\beta + n_1\gamma}{l_1+m_1+n_1}$$

denotes, as we have seen, the point dividing uv in the ratio B : A.

And in general if $\text{U}=0$, $\text{V}=0$, be the tangential equations of any curves, $\text{U}+k\text{V}=0$ denotes a curve touching every line which touches both U and V; for the co-ordinates of every line which satisfies $\text{U}=0$, $\text{V}=0$, must also satisfy $\text{U}+k\text{V}=0$.

The two equations, $u+kv=0$, $u-kv=0$, denote points which

cut the line uv internally and externally in the same ratio; hence $u, v, u+kv, u-kv$, represent four points of a line cut harmonically.

There is no difficulty in seeing (*Conics*, p. 52) that the anharmonic ratio of the system $u, v, u+kv, u+lv, = \frac{k}{l}$; and of the system $u+kv, u+lv, u+mv, u+nv, = \frac{(k-m)(l-n)}{(k-n)(l-m)}$.

We have seen that $\alpha + \beta = 0$ denotes the middle point of the line AB, $\alpha - \beta = 0$ will then denote the point at infinity on the same line; in like manner $\alpha = \gamma$ denotes the point at infinity on the line AC. And $\alpha = \beta = \gamma$ are the co-ordinates of the line at infinity, as is also evident from the consideration that all finite points are equidistant from the line at infinity. $l\alpha + m\beta + n\gamma = 0$ will denote a point at infinity if $l + m + n = 0$; since then the equation of the point is satisfied by the co-ordinates of the line at infinity.

It appears from what has been said, that

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0,$$

denote three points which lie in one right line. This equation gives at once the well-known theorem (*Conics*, p. 34), that if any line cut the sides of a triangle in the points L, M, N, then the continued product of the ratios AM : MC, CL : LB, BN : NA; = 1.

We see hence that the very same equations which in the method of trilinear co-ordinates prove that three lines meet in a point, interpreted in the present method, prove that three points are in one right line. Thus the equations

$$\alpha + \beta = 0, \quad \beta - \gamma = 0, \quad \gamma + \alpha = 0,$$

interpreted one way, express that two external bisectors of the angles of a triangle meet on the internal bisector of the third angle : interpreted the other way, express that the line joining the middle points of two sides of a triangle is parallel to the third side.

5. We give now some examples of the use of these equations of the first degree.

Ex. 1. *The bisectors of the sides of a triangle meet in a point;* viz., in the point $\alpha + \beta + \gamma = 0$.

For this is a point satisfied by $\alpha + \beta = 0$, $\gamma = 0$, the co-ordinates

of the line joining the middle point of AB to C, or by $\alpha+\gamma=0$, $\beta=0$, or by $\beta+\gamma=0$, $\alpha=0$, which are the co-ordinates of the other two bisectors; $\alpha+\beta+\gamma=0$ is then the equation of the centre of gravity of the triangle.

Ex. 2. *The three bisectors of the angles of a triangle meet in a point.*

For since the bisector of the vertical angle cuts the base in the ratio of the sides, the equation of the point where it meets the base may be written either $a\alpha+b\beta=0$ or $\alpha\sin A+\beta\sin B=0$; and the equation $\alpha\sin A+\beta\sin B+\gamma\sin C=0$ is evidently satisfied by the co-ordinates of the line joining this point to the vertex γ. And the symmetry of the equation shows that it is also satisfied by the other bisectors.

Ex. 3. *The three perpendiculars of a triangle meet in a point.*

Its equation is readily seen to be

$$\alpha\tan A+\beta\tan B+\gamma\tan C=0.$$

Ex. 4. *The middle points of the diagonals of a complete quadrilateral lie in one right line.*

Let the equations of three vertices be $\alpha=0$, $\beta=0$, $\gamma=0$, and of the fourth, $l\alpha+m\beta+n\gamma=0$.

The equation of the middle point of one diagonal is $\alpha+\gamma=0$. Of the second is

$$\beta+\frac{l\alpha+m\beta+n\gamma}{l+m+n}=0, \text{ or } (l+m+n)\beta+(l\alpha+m\beta+n\gamma)=0.$$

The equation of the intersection of the opposite sides, AB, CD, is $l\alpha+m\beta=0$; and of AD, BC, is $m\beta+n\gamma=0$.

The equation of the middle point of the line joining these two points is then

$$\frac{l\alpha+m\beta}{l+m}+\frac{m\beta+n\gamma}{m+n}=0, \text{ or } (m+n)(l\alpha+m\beta)+(l+m)(m\beta+n\gamma)=0,$$

which may be written

$$ln(\alpha+\gamma)+m\{(l+m+n)\beta+l\alpha+m\beta+n\gamma\}=0.$$

6. In like manner the same method may be applied to conic sections. The equation $\alpha\beta=k.\gamma\delta$ denotes a conic touching the four sides of the quadrilateral whose vertices are α, β, γ, δ. $\alpha\beta=k\gamma^2$ denotes a conic passing through the points α, β, and

having the lines joining these points to γ for tangents. For as, in former systems, a tangent was defined as a line which meets the curve in two consecutive points, so in the present system a point on the curve is recognised as the intersection of two consecutive tangents; but evidently the two tangents which can be drawn from α to the curve whose equation has been just written, both coincide with the line joining α to γ. And so in general, if $\alpha\phi = \gamma^2\psi$ be the equation of a curve, α is a point on it, and the line joining it to γ the tangent. All the methods of *Conics* (Art. 271, &c.) are directly applicable also to this method. For instance $\mu\alpha - \gamma$, $\beta - \mu\gamma$, denote any tangent to the conic $\alpha\beta = \gamma^2$; and $\mu^2\alpha - 2\mu\gamma + \beta = 0$ denotes the point of contact. And generally if the equation of any point involve an indeterminate in the second degree, the locus of that point will be a conic, whose equation is found by forming the condition that the equation in μ should have equal roots. It seems scarcely necessary to add examples of questions solved by this method, since any of the examples solved at *Conics*, p. 233, may, without alteration of the algebraical work, be taken as an example of this method, by merely altering the interpretation of the equations. For instance, Example 1 is the solution of the question, "A triangle is inscribed in a conic, and two sides pass through fixed points, to find the envelope of the third." We therefore add but one more example.

To find the locus of the point which cuts in a given ratio the intercept of a variable tangent to a conic between two fixed tangents.

Let γ be the intersection of the fixed tangents, and the equation of the conic $\alpha\beta = k^2\gamma^2$; then $\mu\alpha = k\gamma$ denotes the point where any variable tangent meets the tangent joining $\alpha\gamma$; $\beta = \mu k\gamma$ is the point where the same tangent meets $\beta\gamma$; and

$$\frac{A}{\mu - k}(\mu\alpha - k\gamma) + \frac{B}{1 - \mu k}(\beta - \mu k\gamma) = 0$$

is the point dividing in the given ratio B : A the line joining these points. Clearing of fractions and arranging, the equation becomes

$$(Ak\alpha + Bk\gamma)\mu^2 - \{A\alpha + B\beta + (A + B)k^2\gamma\}\mu + Ak\gamma + Bk\beta = 0,$$

a point whose locus is

$$4k^2.(A\alpha + B\gamma)(A\gamma + B\beta) = \{A\alpha + B\beta + (A + B)k^2\gamma\}^2.$$

7. In the method of trilinear co-ordinates it is not possible to tell whether a curve be an ellipse, hyperbola, or parabola, unless the angles of the triangle $\alpha\beta\gamma$ be given. In the present method, however, a test can readily be found. *The curve will be a parabola,* if the co-ordinates of the line at infinity, $\alpha = \beta = \gamma$, satisfy the equation; and since the equation is homogeneous, this will be the case *if the sum of the coefficients* $= 0$. For instance, $\alpha\beta = \gamma^2$ represents a parabola, and we see that the product of the perpendiculars from two points of the curve on any tangent is *equal* to the square of the perpendicular from the pole of the line joining these points.

To examine whether the curve

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 + 2D\alpha\gamma + 2E\beta\gamma + F\gamma^2 = 0$$

be an ellipse or hyperbola, we should examine whether the line at infinity meets the curve in two real points or not. The condition that the point $l\alpha + m\beta + n\gamma = 0$ should be on the curve is found by the same process as in the trilinear methods is found the condition that a given line should touch the curve. It is, therefore (see *Conics*, p. 328),

$$(E^2 - CF)\, l^2 + (D^2 - AF)\, m^2 + (B^2 - AC)\, n^2 + 2(AE - BD)\, mn \\ + 2\,(CD - BE)\, nl + 2\,(BF - DE)\, lm = 0.$$

Now the general equation of a point at infinity is

$$l\alpha + m\beta = (l + m)\,\gamma.$$

If, therefore, in the previous condition we put $n = -(l + m)$, and solve for $l : m$, we should have the equation of the points where the line at infinity meets the curve. And if we form the condition that this equation in $l : m$ should have real roots, we shall have the condition that the curve should be an hyperbola. This will be found to be simply that the sum of the coefficients,

$$A + 2B + C + 2D + 2E + F,$$

and the well-known function,

$$AE^2 + CD^2 + FB^2 - ACF - 2BDE,$$

should have the same sign. If they have different signs the curve will be an ellipse. If the latter function $= 0$, the equation is plainly resolvable into factors, and represents two points.

8. The equation of the conic inscribed in a triangle is

$$l\alpha\beta + m\beta\gamma + n\gamma\alpha = 0;$$

and the methods by which this equation is treated (*Conics*, p. 247) apply here, the interpretation of the equations being suitably altered. From our knowledge of the segments made by the inscribed circle on the sides of a triangle we can readily perceive that the equation of the inscribed *circle* is

$$(s-a)\,\beta\gamma + (s-b)\,\gamma\alpha + (s-c)\,\alpha\beta = 0,$$

or, what is the same,

$$\beta\gamma\cot\tfrac{1}{2}A + \gamma\alpha\cot\tfrac{1}{2}B + \alpha\beta\cot\tfrac{1}{2}C = 0.$$

The equation, we have seen, will represent a parabola if the sum of the coefficients $= 0$, so that $\dfrac{l}{\alpha} + \dfrac{m}{\beta} = \dfrac{l+n}{\gamma}$ is the general equation of a parabola touching the sides of the triangle $\alpha\beta\gamma$.

The equation of the conic circumscribing a triangle is

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

or $$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2lm\alpha\beta - 2mn\beta\gamma - 2nl\gamma\alpha = 0;$$

and $l\alpha - m\beta = 0$ is the equation of the point where the tangent at the vertex γ meets the opposite side. Hence, from examining the segments made by the tangent to the circumscribing *circle* on any side, we can see that its equation is

$$\sin A\sqrt{\alpha} + \sin B\sqrt{\beta} + \sin C\sqrt{\gamma} = 0.$$

The pole of any right line $\alpha'\beta'\gamma'$ is as in trilinear co-ordinates (*Conics*, p. 247),

$$\alpha'\frac{dS}{d\alpha} + \beta'\frac{dS}{d\beta} + \gamma'\frac{dS}{d\gamma} = 0.$$

Thus the equation of the centre, which is the pole of the line at infinity ($\alpha = \beta = \gamma$), is

$$\frac{dS}{d\alpha} + \frac{dS}{d\beta} + \frac{dS}{d\gamma} = 0.$$

For instance, the conic $\alpha\gamma = k\beta\delta$, touching the four sides of the quadrilateral whose vertices are $\alpha, \beta, \gamma, \delta$, will have for its centre

$$\alpha + \gamma = k(\beta + \delta),$$

as will appear by writing for δ its value, $l\alpha + m\beta + n\gamma$, where $l + m + n = 1$. The centre then must lie on the line joining the middle points of the diagonals of the quadrilateral.

9. In what precedes we have used exclusively homogeneous equations between α, β, γ; we wish now to show how to interpret such an equation as $\phi(\alpha, \beta, \gamma) = \text{constant}$. In the case of trilinear co-ordinates we were able to do this by the help of the relation which subsists between the three perpendiculars from any point on the sides of a triangle, viz., $\alpha \sin A + \beta \sin B + \gamma \sin C = \text{const.}$

We proceed, therefore, now to inquire what relation exists between the three perpendiculars from the vertices of any triangle on any variable right line.

We employ Cartesian co-ordinates: we might simplify our equations by a particular choice of axes, but, for the sake of symmetry, take them in the most general form. Let any line be $x \cos\theta + y \sin\theta + p = 0$, then

$$\begin{aligned} x_{\prime} \cos\theta + y_{\prime} \sin\theta + p &= \alpha, \\ x_{\prime\prime} \cos\theta + y_{\prime\prime} \sin\theta + p &= \beta, \\ x_{\prime\prime\prime} \cos\theta + y_{\prime\prime\prime} \sin\theta + p &= \gamma, \end{aligned}$$

and from these equations we want to eliminate θ and p. Multiply the first by $y_{\prime\prime} - y_{\prime\prime\prime}$, the second by $y_{\prime\prime\prime} - y_{\prime}$, the third by $y_{\prime} - y_{\prime\prime}$, and we have

$$M \, . \cos\theta = \alpha(y_{\prime\prime} - y_{\prime\prime\prime}) + \beta(y_{\prime\prime\prime} - y_{\prime}) + \gamma(y_{\prime} - y_{\prime\prime});$$

where $M = x_{\prime}(y_{\prime\prime} - y_{\prime\prime\prime}) + x_{\prime\prime}(y_{\prime\prime\prime} - y_{\prime}) + x_{\prime\prime\prime}(y_{\prime} - y_{\prime\prime})$ = double the area of the triangle. Similarly

$$- M \sin\theta = \alpha(x_{\prime\prime} - x_{\prime\prime\prime}) + \beta(x_{\prime\prime\prime} - x_{\prime}) + \gamma(x_{\prime} - x_{\prime\prime}).$$

θ is eliminated by squaring and adding these equations.

Let $(x_{\prime\prime} - x_{\prime\prime\prime}) = a \cos\phi$, $y_{\prime\prime} - y_{\prime\prime\prime} = a \sin\phi$, where ϕ is the angle which the side a makes with the axis of x; and let $M = ap_{\prime}$, where $p_{\prime}$ is the perpendicular on the side a from the opposite angle; the result then becomes

$$\frac{\alpha^2}{p_{\prime}^2} + \frac{\beta^2}{p_{\prime\prime}^2} + \frac{\gamma^2}{p_{\prime\prime\prime}^2} - \frac{2\alpha\beta \cos C}{p_{\prime}p_{\prime\prime}} - \frac{2\beta\gamma \cos A}{p_{\prime\prime}p_{\prime\prime\prime}} - \frac{2\gamma\alpha \cos B}{p_{\prime\prime\prime}p_{\prime}} = 1,$$

which, for shortness, we shall write $\Omega = 1$.

If then we are given an equation $\phi(\alpha\beta\gamma) = \text{const.}$ we can, by the help of this equation, bring it to a homogeneous function of $\alpha\beta\gamma$. For we can introduce a factor z, so as to make the equation homogeneous, and eliminate z by the help of the equation

$$z^2 = \Omega.$$

Thus the equation of the circle whose centre is $l\alpha + m\beta + n\gamma$, and radius is r, is evidently $\frac{l\alpha + m\beta + n\gamma}{l+m+n} = r$, which is therefore reduced to

$$(l\alpha + m\beta + n\gamma)^2 = r^2(l + m + n)^2\,\Omega.$$

And, in general, if the equation $\phi(\alpha, \beta, \gamma, \text{const.}) = 0$ be of the n^{th} degree, and involve the constant only in the even powers, the curve is of the n^{th} class; but if any odd powers enter, since we have to square in order to bring it to a homogeneous function of α, β, γ, the curve is of the $2n^{th}$ class.

Thus the equation $\alpha\beta\gamma\delta = \text{const.}$ denotes a curve of the fourth class, being equivalent to $\alpha\beta\gamma\delta = \Omega^2$; but $\alpha\beta\gamma = \text{const.}$ denotes a curve of the sixth class, being equivalent to $\alpha^2\beta^2\gamma^2 = \Omega^3$.

10. In trilinear co-ordinates we found that the paradoxical equation $C = 0$ denoted the right line at infinity. It is then natural to inquire what is the meaning of the equation

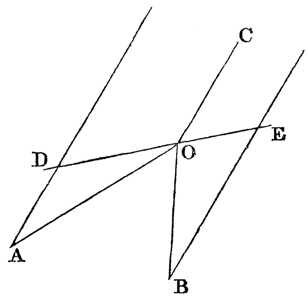
$$\frac{\alpha^2}{p_{\prime}^2} + \frac{\beta^2}{p_{\prime\prime}^2} + \frac{\gamma^2}{p_{\prime\prime\prime}^2} - \frac{2\alpha\beta\cos C}{p_{\prime}p_{\prime\prime}} - \frac{2\beta\gamma\cos A}{p_{\prime\prime}p_{\prime\prime\prime}} - \frac{2\gamma\alpha\cos B}{p_{\prime\prime\prime}p_{\prime}} = 0,$$

which appears to denote a curve of the second class. And first it appears from *Conics*, p. 247, that the equation is resolvable into factors, and therefore denotes two points. Moreover it is easy to verify that the sum of the coefficients of the equation vanishes, and that therefore these points are at infinity. And what the points are we can readily see from the last Article; for denoting, for brevity, the two factors by ω, ω', we have proved that the equation of any circle whose centre is δ is of the form $\delta^2 = k^2\omega\omega'$. It follows then from Art. 6 that ω, ω' represent the two points of contact of the tangents which can be drawn from δ; that is, they are the two points at infinity on any circle. What is meant, then, in this system of co-ordinates, by the equation, constant $= 0$, is the two points at infinity on any circle.

In like manner, if α, β be the two foci of a conic; expressing the property that the product of the perpendiculars from these points on any tangent is constant, we find the equation of the curve to be of the form $\alpha\beta = k\omega\omega'$, which at once gives the theorem that all confocal conics may be considered as inscribed in the same quadrilateral.

11. Enough has been said to show how these three-point co-ordinates are to be used. We shall now show that this system includes some other co-ordinate methods at first sight very different. We shall begin by inquiring what the meaning of the equations will be when one of the points of reference γ is at infinity. We have proved (Art. 3) that the equation of any point O is in general

$$\frac{\sin BOC}{OA}\alpha + \frac{\sin COA}{OB}\beta + \frac{\sin AOB}{OC}\gamma = 0.$$



When C goes off to infinity, both γ and OC become infinite, but their ratio remains finite, and = sin COE, where DOE is any line drawn through the point O. The equation then becomes $\frac{\sin BOC}{OA}\cdot\frac{\alpha}{\sin COE}$

$$+\frac{\sin COA}{OB}\cdot\frac{\beta}{\sin COE} + \sin AOB = 0.$$

In this equation $\frac{\sin BOC}{OA}$, $\frac{\sin COA}{OB}$, sin AOB, are constants, when O is given; $\frac{\alpha}{\sin COE}$, $\frac{\beta}{\sin COE}$ are variable; but since sin COE = sin ODA, we have $\frac{\alpha}{\sin COE} = AD$, $\frac{\beta}{\sin COE} = BE$. If then we take as co-ordinates AD, BE, the intercepts made by a variable line on two fixed parallel lines, we see that any linear equation,

$$Ax + By + C = 0,$$

denotes a point; and that this equation may be considered as the particular form assumed by the homogeneous equation,

$$Ax + By + Cz = 0,$$

when the point z is at infinity.

It follows then that any relation of the n^{th} degree between x and y denotes a curve of the n^{th} class; for we obviously get an equation of the n^{th} degree to determine the tangents which can be drawn from any point $Ax + By + C$ to the curve.

12. If, in the equation of any point O, according to this system, we substitute the co-ordinates of any line, the quantity

$Ax' + By' + C$ will be proportional to the intercept made by the line on a parallel to AD drawn through O. For let the co-ordinates of a parallel line through O be x'', y'', and $Ax'' + By'' + C$ identically vanishes; therefore

$$Ax' + By' + C = A(x' - x'') + B(y' - y'');$$

but $x' - x''$ and $y' - y''$ are each equal to the intercept in question. We add an example or two to illustrate the use of co-ordinates of this kind. We know from the theory of conic sections, that the general equation of the second degree can be reduced to the form $\alpha\beta = k^2$; where α, β are certain linear functions of the co-ordinates. This is an analytical fact wholly independent of the interpretation we give the equations. It follows then, that the most general equation of curves of the second class in this system of co-ordinates can be reduced to the same form, $\alpha\beta = k^2$; but this denotes a curve on which the points α, β lie; and which has for tangents at these points the parallel lines joining these points to the infinitely distant point k; we have then the well-known theorem, that "any variable tangent to a conic intercepts on two fixed parallel tangents portions whose rectangle is constant."

We shall have occasion to prove afterwards that the general equation of the third degree can always be reduced to the form $\alpha\beta\gamma = k^2\delta$; in this system this denotes a curve of the third class, having the points α, β, γ on the curve, having for tangents at those points the parallel lines joining these points to the infinitely distant point k; from each of the points α, β, γ can be drawn a third tangent to the curve, and these three tangents meet in a point δ. We learn, then, that "any variable tangent to a curve of the third class intercepts on three parallel tangents portions whose continued product is in a constant ratio to the intercept on a parallel line drawn through the point δ, where the tangents from α, β, γ intersect."

13. We are led again to quite a different system of co-ordinates, if we suppose two of the points of reference at infinity.

The equation

$$\frac{\sin BOC}{OA}\alpha + \frac{\sin COA}{OB}\beta + \frac{\sin AOB}{OC}\gamma = 0$$

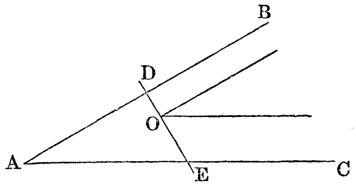
becomes, as in Art. 11,

$\sin BOC . \sin DOA + \sin COA . \sin BOD + \sin AOB . \sin COE = 0$;

or since $$\frac{\sin BOD}{\sin DOA} = \frac{OA}{AD}, \quad \frac{\sin COE}{\sin EOA} = \frac{OA}{AE},$$

$$\sin BOC \frac{1}{OA} + \sin COA \frac{1}{AD} + \sin AOB \frac{1}{AE} = 0.$$

The point O being supposed fixed, the only things variable in this equation are $\frac{1}{AD}, \frac{1}{AE}$; if then we take as co-ordinates *the reciprocals* of the intercepts made by any line on two fixed intersecting lines, any linear relation $A + Bx + Cy = 0$ between these co-ordinates denotes that the line passes through a fixed point, and may be considered as the form assumed by the equation of a point when two of the points of reference are at infinity. So in like manner any relation of the n^{th} degree between those co-ordinates denotes a curve of the n^{th} class; and if the equation be rendered homogeneous by the introduction of a factor z, z denotes the intersection of the axes, and x and y the points at infinity on these lines.



Thus the equation $xy = \text{const.}$, or $xy = z^2$, denotes a curve passing through the points x, y, and having the lines joining these points to z for tangents; that is, an hyperbola whose asymptotes are the axes, as is otherwise evident.

14. We have commenced by an explanation of these systems of tangential co-ordinates, because the principle of duality is one which we wish the reader to bear in mind in his whole perusal of the following pages. We shall generally use trilinear or Cartesian equations, but the reader may, if he pleases, suppose the equations to belong to any of the tangential systems here explained, and will then derive from them theorems the reciprocals of those which we shall deduce from them. We might, it is true, have reminded the reader of the possibility of reciprocating every theorem (as was shown, *Conics*, p. 251, &c.), but we prefer the point of view here explained, in which the one theorem is not deduced from the other, but both are regarded as alike entitled to be considered the geometric interpretation of the same equations.

For instance, the system $S - \alpha\beta = 0$, $S - \alpha\gamma = 0$, $\alpha(\beta - \gamma) = 0$, may be interpreted either, "that if three conics have two points common, their remaining common chords meet in a point;" or if the same equations belong to any system of tangential co-ordinates, that "if three conics have two tangents common, the three remaining intersections of common tangents lie in one right line."

It may, however, be interesting to show how all the results of the present Chapter may be derived by the method of reciprocal polars. Any homogeneous equation of the n^{th} degree between the perpendiculars let fall from a variable point on three fixed lines denotes a curve of the n^{th} degree; take the reciprocal of this with regard to any point O; divide the given equation by OP^n, P being the variable point; for each ratio $\frac{PM}{OP}$, PM being the perpendicular on one of the fixed lines, we may substitute (*Conics*, pp. 95, 258) $\frac{pm}{op}$; pm being the perpendicular from the fixed point p on a variable line. And the denominators being constant, we see that, for the curve of the n^{th} class (the reciprocal of the given curve), there exists a homogeneous relation between the perpendiculars let fall from three fixed points on any variable tangent.

15. But if now the point with regard to which the reciprocals are taken be on one of the three lines of reference α, we should find as before, for the perpendiculars on the other two lines, $\frac{PM}{OP} = \frac{pm}{op}$; but the ratio $\frac{PA}{OP}$ (where PA is the perpendicular on α) expresses the sine of the angle which the radius vector OP makes with a fixed line α through the origin, to which corresponds on the reciprocal figure the cosine of the angle which the variable line makes with the same line; there results then a homogeneous relation of the n^{th} degree between the cosine of this angle and the two perpendiculars let fall from two fixed points on the variable line; or if the whole equation be divided by $\cos^n\theta$, we get a non-homogeneous relation of the n^{th} degree between the two intercepts made by the variable line on lines drawn through two fixed points in a certain fixed direction.

16. Lastly, if the point with regard to which the reciprocals are taken be the intersection of the fixed lines α, β, there results, as before, a homogeneous relation between the cosines of the two angles which the variable line makes with two fixed directions, and the perpendicular let fall on it from a fixed point; or, dividing the whole equation by the n^{th} power of this perpendicular, we obtain a relation of the n^{th} degree between the reciprocals of the intercepts made by the variable line on lines drawn through the fixed point in the given directions.*

CHAPTER II.

ON THE GENERAL PROPERTIES OF CURVES OF THE N^{TH} DEGREE.

SECT. I.—ON THE NUMBER OF TERMS IN THE GENERAL EQUATION.

17. The first step towards obtaining a knowledge of the general properties of curves of the n^{th} degree is the ascertaining the number of terms in the general equation. We should thereby be enabled, on being given any equation of the n^{th} degree, by simply counting the number of independent constants in the equation, to know whether or not the given form were one to which all equations of the n^{th} degree could be reduced. For example, the general equation of the second degree contains five independent constants. If then we were given any other equation of the second degree, containing five constants, for instance,

$$(x-\alpha)^2 + (y-\beta)^2 = (ax+by+c)^2,$$

or $$\{(x-\alpha)^2 + (y-\beta)^2\}^{\frac{1}{2}} + \{(x-\alpha_{,})^2 + (y-\beta_{,})^2\}^{\frac{1}{2}} = c,$$

* The materials for this Chapter are principally taken from Plücker's System der Analytischen Geometrie, Abschnitt I. § 2. The co-ordinate system of Art. 3 seems (as far as I have been able to judge by a hasty glance) to coincide with the method of Mobius's Barycentric Calculus, published in 1827. A sketch of the co-ordinate system of Art. 11 is given by M. Chasles in Quetelet's Correspondence for 1830, vol. vi. p. 81. And the system of Art. 13 is the subject of a tract on Tangential Co-ordinates, published in this country by Dr. Booth in the year 1840. But to M. Plücker, I believe, belongs the merit of having, by his general views respecting point and line co-ordinates, put the principle of duality on its true analytical basis.

we could expand, and comparing the equation (as at *Conics*, p. 69) with the general equation of the second degree, should obtain a sufficient number of equations to determine α, β, &c., in terms of the coefficients of the general equation. We see then that any equation of the second degree may, in general, be reduced to either of the above forms, and we might thus obtain a proof of the properties of the foci and of the directrix. The equation

$$(ax + by + c)^2 = (a'x + b'y + c')\ (a''x + b''y + c'')$$

contains seven independent constants. The problem, therefore, to express these in terms of the coefficients in the general equation, is indeterminate; as is also geometrically evident, since the equation may be thrown into this form by taking

$$a'x + b'y + c',\ \ a''x + b''y + c''$$

to represent any two tangents, and $ax + by + c$, their chord of contact. The equations

$$(ax + by)^2 = cx + dy + e,$$

$$(ax + by + 1)\ (a'x + b'y + 1) = 0,$$

contain each but four independent constants, and must, therefore, implicitly involve one other condition; or, in other words, the general equation cannot be thrown into either of these forms, unless one other condition be fulfilled. This is geometrically evident, since the first equation denotes a parabola, and the second, two right lines. The general equation of a circle,

$$(x - \alpha)^2 + (y - \beta)^2 = r^2,$$

containing but three expressed constants, must involve two others implicitly; or the general equation cannot be thrown into this form unless two conditions be fulfilled. And so again the equation

$$S - kS' = 0,$$

containing but one expressed constant, must imply four other conditions, as we otherwise know, since the conic expressed by this equation passes through four fixed points.

18. Some caution must be used in the application of these principles. Thus the equation

$$(x - \alpha)^2 + (y - \beta)^2 = ax + by + c,$$

appears to contain five constants, and, therefore, to be a form to

which every equation of the second degree is reducible. But if we expand, we shall see that the constants do not enter into the highest terms of the equation, and that there are but three equations available to determine α, β, &c. The equation can, therefore, not be thrown into this form unless two other conditions be fulfilled. In like manner the equation

$$aS_1 + bS_2 + cS_3 + dS_4 + eS_5 + fS_6 = 0,$$

where S_1, &c., are six conics, is a form to which the equation of any conic may be reduced; but suppose three of the equations of these conics to be connected by the relation $S_3 = S_1 + kS_2$; substituting this value, the equation would be found to contain but four independent constants, and the general equation could not be reduced to this form unless some one condition were fulfilled.

19. Having thus endeavoured to give the reader an idea of the nature of the advantage to be gained by a knowledge of the number of terms in the general equation of the n^{th} degree, we proceed to an investigation of this problem. The general equation of the n^{th} degree between two variables may be written,

$$\begin{array}{c} A \\ + Bx + Cy \\ + Dx^2 + Exy + Fy^2 \\ + \dots\dots\dots\dots\dots \\ + Px^n + Qx^{n-1}y + \dots\dots + Rxy^{n-1} + Sy^n = 0. \end{array}$$

And the number of terms in this equation is plainly the sum of the series $1 + 2 + 3 + \dots\dots + (n + 1)$, and is therefore equal to $\frac{(n+1)\ (n+2)}{1\,.\,2}$, as has been already proved (*Conics*, p. 70).

We shall sometimes write the general equation in the abbreviated form,

$$u_0 + u_1 + u_2 + \dots\dots + u_n = 0,$$

where u_0 denotes the absolute term, and u_1, u_2, u_n, &c., denote the terms of the first, second, n^{th}, &c., degrees in x and y.

We shall also sometimes employ the equation in trilinear coordinates, which only differs from that just written in having a third variable z introduced, so as to make the equation homogeneous, viz.,

$$u_0z^n + u_1z^{n-1} + u_2z^{n-2} + \dots\dots + u_{n-1}z + u_n = 0.$$

The number of terms is evidently the same as in the preceding case (*Conics*, p. 225).

20. The number of conditions necessary to determine a curve of the n^{th} degree is one less than the number of terms in the general equation; or is equal to $\frac{n(n+3)}{2}$. For the equation represents the same curve if it be multiplied or divided by any constant; we may therefore divide by A, and the curve is completely determined if we can determine the $\frac{n(n+3)}{2}$ quantities $\frac{B}{A}$, $\frac{C}{A}$, &c.

Thus a curve of the n^{th} degree is in general determined when we are given $\frac{n(n+3)}{2}$ points on it; for the co-ordinates of each point through which the curve passes, substituted in the general equation, give a linear relation between the co-efficients. We have, therefore, $\frac{n(n+3)}{2}$ equations of the first degree to determine the same number of unknown quantities, a problem which admits in general of but one solution. We learn then that a curve of the third degree can be described through nine points, one of the fourth degree through fourteen points, and in general *through* $\frac{n(n+3)}{2}$ *points can be described one, and but one, curve of the* n^{th} *degree.*

21. When we say that $\frac{n(n+3)}{2}$ points determine a curve of the n^{th} degree, we would not be understood to mean that they always determine a *proper* curve of that degree. All that we have proved is, that an equation of the n^{th} degree can be determined which will be satisfied for the given points; but this equation may be the product of two or more others of lower dimensions. Thus, five points in general determine a conic, but if three of them lie on a right line, no system of the second degree can be described through the points, except that formed by this right line and the line joining the other two points. And, in general, it is evident that, if of the $\frac{n(n+3)}{2}$ points more than np lie on a

curve of the p^{th} degree (p being less than n), a *proper* curve of the n^{th} degree cannot be described through the points, for we should then have the absurdity of two curves of the n^{th} and p^{th} degrees intersecting in more than np points (*Conics*, pp. 10, 209). The only system of the n^{th} degree which can be described through such a set of points is the curve of the p^{th} degree, together with a curve of the $n-p^{th}$ through the remaining points.

We may even fix a lower limit to the number of points determining a proper curve of the n^{th} degree which can lie on a curve of the p^{th} degree, and can show that this number cannot be greater than $np - \frac{(p-1)(p-2)}{1.2}$. For if we suppose that one more of the points (viz., $np - \frac{(p-1)(p-2)}{2} + 1$) lie on a curve of the p^{th} degree, subtracting this number from $\frac{n(n+3)}{2}$, it will be found that the number of remaining points is $\frac{(n-p)(n-p+3)}{1 \cdot 2}$, and that therefore a curve of the $(n-p)^{th}$ degree can be described through them. This with the curve of the p^{th} degree forms a system of the n^{th} degree through the points; and it follows from the last Article that it is in general impossible to describe through them any other.

22. There is one case, however, in which the solution of Art. 20 may fail. When we solve m linear equations between m unknown quantities, the solution in general comes out in the form of a fraction (see note on Elimination at the end of the volume). We should have, for instance, a solution of the form

$$\frac{B}{A} = \frac{B_{\prime}}{A_{\prime}}, \quad \frac{C}{A} = \frac{C_{\prime}}{A_{\prime}}, \text{ \&c.}$$

Now it might so happen that the given values of the co-ordinates of the $\frac{n(n+3)}{2}$ points might cause both numerator and denominator of every one of these fractions to vanish. In this case, then, the given points would plainly be insufficient to determine the curve, and through them could be described an infinity of curves of the n^{th} degree. It is not difficult to explain the geometrical reason why such cases should sometimes occur.

Let us, for simplicity, commence with the example of curves of the third degree. Let $U = 0$, $V = 0$, be the equations of two such curves, both passing through eight given points; then the equation of any curve of the third degree passing through these points must be of the form $U - kV = 0$. For this equation, from its form, denotes a curve of the third degree passing through the eight given points, and it contains an arbitrary constant k which can be so determined that the curve shall pass through any ninth point. We should, in fact, have $k = \frac{U'}{V'}$, where U', V' are the results of substituting the co-ordinates of the ninth point in U and V. This gives a determinate value for k in every case but one, viz., when the ninth point lies on both U and V; for since two curves of the m^{th} and n^{th} degrees intersect in mn points, U and V intersect not only in the eight given points, but also in one other. For the co-ordinates of this point k takes the value $\frac{0}{0}$; and indeed the form of the equation sufficiently shows that every curve represented by the equation $U - kV = 0$ passes through *all* the intersections of U and V. Hence we have the important theorem, *All curves of the third degree which pass through eight fixed points pass also through a ninth.* And we can perceive that nine points are not always sufficient to *determine* a curve of the third degree, for that we can describe a curve of the third degree through the intersections of two such curves, and through any tenth point.

23. The same reasoning applies to curves of any degree. If there be given a number of points one less than that which will determine the curve $\left\{\frac{n(n+3)}{2} - 1\right\}$, then $U - kV = 0$ (where U and V are any two particular curves of the system) is the most general equation of curves of the n^{th} degree passing through these points. For the equation contains one arbitrary constant, to which we can assign such a value that the curve shall pass through any remaining point, and be therefore completely determined. But the form of the equation shows that the curve must pass through *all* the n^2 points common to U and V, and therefore not

only through the $\frac{n(n+3)}{2} - 1$ given points, but also through as many more as will make up the entire number to n^2. Hence, *All curves of the n^{th} degree which pass through $\frac{n(n+3)}{2} - 1$ fixed points pass also through $\frac{(n-1)(n-2)}{1 \, . \, 2}$ other fixed points.*

24. The following is a useful deduction from the preceding theorem: *If of the n^2 points of intersection of two curves of the n^{th} degree, np lie on a curve of the p^{th} degree (p being less than n), the remaining $n(n-p)$ will lie on a curve of the $n-p^{th}$ degree.* For describe a curve of the $(n-p)^{th}$ degree through $\frac{(n-p)(n-p+3)}{2}$ of these remaining points, and this, together with the curve of the p^{th} degree, form a system passing through $\frac{(n-p)(n-p+3)}{2} + np$ points; and since this number (being equal to $\frac{n(n+3)}{2} - 1 + \frac{(p-1)(p-2)}{2}$) cannot be less than $\frac{n(n+3)}{2} - 1$, this system will be certain to pass through all the remaining points.

It is to be understood in these theorems concerning the intersections of curves of the n^{th} degree, that the curves need not be proper curves of that degree, for the demonstration in Art. 23 holds equally even though U or V be resolvable into factors. As an illustration of the theorem of this Article, we add the following: *If a polygon of $2n$ sides be inscribed in a conic, the $n(n-2)$ points where each odd side intersects the non-adjacent even sides will lie in a curve of the $n-2^{nd}$ degree.* For the product of all the odd sides forms one system of the n^{th} degree, and the product of all the even sides another; these systems intersect in n^2 points, viz., since each odd side has two adjacent and $n-2$ non-adjacent even sides, in the $2n$ vertices of the polygon, and the $n(n-2)$ points, which are the subject of the present theorem. But since, by hypothesis, the $2n$ vertices lie on a conic, the remaining $n(n-2)$ points, by this Article, lie on a curve of the $n-2^{nd}$ degree.

25. Pascal's theorem is a particular case of the theorem just

given, but on account of the importance that the learner should clearly understand the principle of the foregoing demonstrations, we think it advisable to repeat in other words the proof already given.

Denote the equations of the sides of the hexagon by the first six letters of the alphabet; then $ACE - kBDF = 0$ is the equation of a system of curves of the third degree passing through AB, BC, CD, DE, EF, FA, and also through AD, BE, CF. If the first six points lie on a conic S, then the curve of the system determined by the condition that it shall pass through any seventh point of the conic S must give $ACE - k'BDF = SL$. For it cannot be a proper curve of the third degree, since no such curve can have more than six points common with S. The right line L will therefore contain the three points AD, BE, CF.

We may add, that it is this proof of Pascal's theorem which leads most readily to Steiner's and Kirkman's theorems (*Conics*, p. 321). Thus, let

$$12 \cdot 34 \cdot 56 - 45 \cdot 61 \cdot 23 = SL,$$

where 12 denotes the line joining the vertices 1, 2, &c.; and where L consequently denotes the line through the intersections of the opposite sides, 12, 45; 34, 61; 56, 23;

and let
$$12 \cdot 34 \cdot 56 - 36 \cdot 25 \cdot 14 = SM;$$

then obviously

$$45 \cdot 61 \cdot 23 - 36 \cdot 25 \cdot 14 = S \cdot (M - L);$$

or the Pascal line indicated by the latter equation passes through the intersection of the other two.

26. It has been proved that, although two curves of the n^{th} degree intersect in n^2 points, yet n^2 points, taken arbitrarily, will not be the intersections of two such curves; but that $n^2 - \frac{(n-1)(n-2)}{2}$ of them being given, the rest will be determined. A similar theorem holds with regard to the np points of intersection of two curves of the n^{th} and p^{th} degrees. Thus though a curve of the third degree intersects one of the fourth in twelve points, yet through twelve points taken arbitrarily on a curve of the third degree, it will, in general, be impossible to describe a proper curve of the fourth degree. For the system of the fourth degree through

these twelve and any other two points will in general be no other than the curve of the third degree and the line joining the two points. And, generally, *Every curve of the* n^{th} *degree which is drawn through* $np - \frac{(p-1)(p-2)}{2}$ *points on a curve of the* p^{th} *degree* (p *being less than* n) *meets this curve in* $\frac{(p-1)(p-2)}{2}$ *other fixed points.* For we had occasion in Art. 24 to see that

$$np - \frac{(p-1)(p-2)}{2} + \frac{(n-p)(n-p+3)}{2} = \frac{n(n+3)}{2} - 1;$$

therefore, by Art. 23, every system of the n^{th} degree described through the given points, and $\frac{(n-p)(n-p+3)}{2}$ others, passes through $\frac{(n-1)(n-2)}{2}$ other fixed points. But one system of the n^{th} degree which can be described through the points is the given curve of the p^{th} degree and one of the $(n-p)^{th}$ through the additional assumed points. The $\frac{(n-1)(n-2)}{2}$ new points must therefore lie, some on one, some on the other of these two curves. And it is evident that these points must be so distributed between them as to make up the total number of points, in the first case, to np, in the second to $n(n-p)$. Hence the truth of the theorem enunciated is manifest.

27. A further extension of this theorem has been given by Mr. Cayley: "*Any curve of the* r^{th} *degree* (r *being greater than* m *or* n, *but not greater than* $m + n - 3$), *which passes through all but* $\frac{(m+n-r-1)(m+n-r-2)}{2}$ *of the* mn *intersections of two curves of the* m^{th} *and* n^{th} *degree, will pass also through the remaining intersections.*"

The reader will more easily understand the spirit of the general proof we are about to give, by applying it first to a particular example. "Any curve of the fifth degree which passes through fifteen of the intersections of two curves of the fourth degree will also pass through the remaining intersection." For take two arbitrary points on each of the curves of the fourth degree. These four, with the fifteen given points, make nineteen points, through

which, if several curves of the fifth degree pass, they will (by Art. 23) pass through six other fixed points. But each curve of the fourth degree, together with the line joining the two arbitrary points on the other curve, forms a system of the fifth degree through the nineteen points. Hence *all* the intersections of the given curves of the fourth degree lie on every curve of the fifth degree through the points. Q. E. D.

So, in general, take $\frac{(r-m)(r-m+3)}{2}$ arbitrary points on the curve of the n^{th} degree, and through them draw a curve of the $(r-m)^{th}$ degree; and take $\frac{(r-n)(r-n+3)}{2}$ points on the curve of the m^{th} degree, and through them draw a curve of the $(r-n)^{th}$ degree; take as many of the mn points of intersection as with the arbitrary points make up $\frac{r(r+3)}{2}-1$: then since the curves of the $(r-m)^{th}$ and m^{th} degree make one system of the r^{th} degree through the points, and the curves of the $(r-n)^{th}$ and n^{th} make another, the intersection of these two systems will be common to every curve of the r^{th} degree through the points. But

$$\frac{r(r+3)}{2}-1-\frac{(r-m)(r-m+3)}{2}-\frac{(r-n)(r-n+3)}{2}$$
$$=mn-\frac{(m+n-r-1)(m+n-r-2)}{2},$$

as the reader may verify without difficulty. Hence the truth of the theorem appears. This proof would plainly not apply if r be not at least equal to the greatest of m or n; and it would also be inapplicable if $r-m$ were not less than n, since otherwise it would not be possible to describe, through the assumed points on the curve of the n^{th} degree, a curve of the $(r-m)^{th}$ degree, distinct from the curve of the n^{th} degree.*

* Euler appears first to have noticed the paradox, that two curves of the n^{th} degree may intersect in a greater number of points than are sufficient to determine such a curve (see a memoir in the Berlin Transactions for 1748, "On an apparent Contradiction in the Theory of Curves"). The same difficulty is pointed out by Cramer, in his "Introduction à l'Analyse des Lignes courbes algébriques," published in the year 1750. It was only comparatively recently, however, that the important geometrical theorems were observed,

SECT. II.—ON THE NATURE OF THE MULTIPLE POINTS AND TANGENTS OF CURVES.

28. The simplest method of introducing to the reader the subject of the singular points and lines connected with curves seems to be, first, to illustrate by particular examples the nature of these points and lines, and afterwards to lay down rules by which their existence may be detected in general.

We shall employ the Cartesian equation given in Art. 19. If we transform this equation to polar co-ordinates, by substituting $\rho\cos\theta$, $\rho\sin\theta$ for x and y (or if the axes be not rectangular, $m\rho$, $n\rho$, as at *Conics*, p. 121), we get an equation of the n^{th} degree in ρ, whose roots are the distances from the origin of the n points, where the curve is met by a line drawn through the origin, making an angle θ with the axis of x.

29. If in the general equation the absolute term $A = 0$, then the origin is a point on the curve; for the equation is evidently satisfied by the values $x = 0$, $y = 0$, that is, by the co-ordinates of the origin.

The same thing appears from the equation expressed in polar co-ordinates,

$$(B\cos\theta + C\sin\theta)\rho + (D\cos^2\theta + E\cos\theta\sin\theta + F\sin^2\theta)\rho^2 + \&c. = 0;$$

for this equation being divisible by ρ, one of its roots must be $\rho = 0$, whatever be the value of θ, and therefore one of the n points, in which every line drawn through the origin meets the curve, will in this case coincide with the origin itself.

The other $(n-1)$ points will in general be distinct from the

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which are derived from this principle. In the year 1827 M. Gergonne gave the theorem of Art. 24 (Annales, vol. xvii. p. 220). The general theorem of Art. 23 was given about the same time by M. Plücker (Entwickelungen, vol. i. p. 228; and Gergonne's Annales, vol. xix. pp. 97, 129). It was some years afterwards that the cases were discussed of the relation which exists between the points of intersection of curves and surfaces of different degrees (as in Art. 26). These cases were discussed in two papers sent at the same time for publication in Crelle's Journal, one by M. Jacobi (vol. xv. p. 285), the other by M. Plücker (vol. xvi. p. 47). Besides the papers just mentioned, the reader may also consult a memoir by Mr. Cayley (Cambridge Math. Journal, vol. iii. p. 211). The historical sketch given in the present note is taken from Plücker's Theorie der Algebraischen Curven, p. 13.

origin; there is, however, one value of θ, for which a second point will coincide with the origin, viz., if θ be such that

$$B \cos\theta + C \sin\theta = 0.$$

The equation then, becoming

$$(D\cos^2\theta + E\sin\theta\cos\theta + F\sin^2\theta)\,\rho^2 + \&c. = 0,$$

is divisible by ρ^2, and has, therefore, for two of its roots, $\rho = 0$. The line, therefore, answering to this value of θ, meets the curve in two coincident points, or (*Conics*, p. 75) is the *tangent* at the origin.

Since we have a simple equation to determine $\tan\theta$, we see that at a given point on a curve there can, in general, be drawn but one tangent. Its equation is evidently

$$\rho\,(B\cos\theta + C\sin\theta) = 0, \text{ or } Bx + Cy = 0.$$

Hence if the equation of a curve be $u_1 + u_2 + \&c. = 0$ (*the origin being a point on the curve*), *then* $u_1 = 0$ *is the equation of the tangent.*

If $B = 0$, the axis of x is a tangent; if $C = 0$, the axis of y.

30. Let us now, however, suppose that A, B, C are all $= 0$; the coefficients of ρ will then $= 0$, whatever be the value of θ; in this case, therefore, *every* right line drawn through the origin meets the curve in two points which coincide with the origin. The origin is then said to be a *double* point.

We may see now, exactly as in the last Article, that it is in this case possible to draw through the origin lines which meet the curve in three coincident points. For let θ be such as to render the coefficient of $\rho^2 = 0$, or $D\cos^2\theta + E\sin\theta\cos\theta + F\sin^2\theta = 0$, then the equation becomes divisible by ρ^3, and three values of ρ are $= 0$. Since we have a quadratic to determine $\tan\theta$, it follows that there can be drawn through a double point *two* right lines, each of which meets the curve in three coincident points; their equation is

$$\rho^2\,(D\cos^2\theta + E\sin\theta\cos\theta + F\sin^2\theta) = 0, \text{ or } Dx^2 + Exy + Fy^2 = 0.$$

We learn hence that although every line through a double point may, in one sense, be said to be a tangent (since every such line meets the curve in two coincident points), yet that there are two of these lines whose contact is closer than that of the rest: so that it is usual to say that at a double point on a curve there can

be drawn two tangents. If the equation of the curve (the origin being a double point) be written $u_2 + u_3 + \&c. = 0$, then $u_2 = 0$ is the equation of the pair of tangents at the origin.

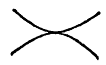
31. It is necessary to distinguish three species of double points, according as the lines represented by u_2 are real, coincident, or imaginary.

I. In the first case the tangents are both real; the double point or *node* is such as that represented in the second figure (Art. 32), arising from the intersection of two branches of the curve, each of which has its own tangent.

A simple illustration of such double points occurs when the given equation is the product of two equations of lower dimensions, or $U = PQ$. The equation $U = 0$ then represents the two curves denoted by $P = 0$ and $Q = 0$. But if these two be considered as making up a complex curve of the n^{th} degree, this curve must be said to have pq double points (the points, namely, where P intersects Q); and at each of these points there are evidently two tangents (viz., the tangents to P and Q).

II. The equation u_2 may be a perfect square; in this case the tangents at the double point coincide, and the curve takes the form represented in the fourth figure (Art. 32). Such points are called *cusps*. They are also sometimes called *stationary points;* for if we imagine the curve to be generated by the motion of a point, at every such cusp the motion in one direction is brought to a stop, and is exchanged for a motion in the opposite direction.

The reader might suppose that we could illustrate these points, as in the last paragraph, by supposing the curve U to break up into two, P and Q, which touch; for every point of contact will be a double point, the tangents at which coincide. But such a point must be classed among singularities of a higher order than those which we are now considering; for the tangent at it meets the complex curve in four consecutive points, viz., two on each of the simple curves, while at the cusps we are considering we have seen that the tangent generally meets the curve in only three consecutive points. In order that the tangent at a cusp should meet the curve in four consecutive points, it is necessary



not merely that u_2 should be a perfect square, but further, that its square root should be a factor in u_3: that is to say, that the equation should be of the form

$$v_1^2 + v_1 v_2 + u_4 + \&c. = 0.$$

Such points arise from the union of two double points, as the reader will readily perceive from the example which we have already given: for when the curves P and Q touch, the point of contact takes the place of two points of intersection.

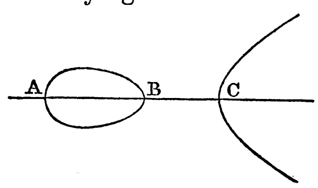
III. The equation $u_2 = 0$ may have both its roots imaginary.

In this case no real point is consecutive to the origin, which is then called a *conjugate point.* Its co-ordinates satisfy the equation of the curve, but it does not appear to lie on the curve, and, in fact, the existence of such points can only be made manifest geometrically by showing that there are points, no line through which can meet the curve in more than $n - 2$ points.

32. As the learner may probably find some difficulty in conceiving the relation of conjugate points to the curve, we shall illustrate the subject by the following example. Let us take the curve,

$$y^2 = (x - a)(x - b)(x - c),$$

where a is less, and c greater than b. This curve is evidently symmetrical on both sides of the axis of x, since every value of x gives equal and opposite values to y. The curve meets the axis of x at the three points $x = a$, $x = b$, $x = c$. When x is less than a, y^2 is negative, and therefore y imaginary: y^2 becomes positive for values of x between a and b; negative again for values between b and c; and, finally, positive for all values of x exceeding c. The curve therefore consists of an oval lying between A and B, and a branch commencing at C, and extending indefinitely beyond it.



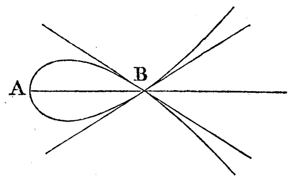
Let us now suppose $b = c$, and the equation will become

$$y^2 = (x - a)(x - b)^2,$$

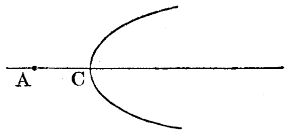
where b is greater than a. The point B has now closed up to C; the oval has joined the infinite branch, and the point B has become a double point. (See first figure on next page.)

But, on the other hand, let $b = a$. Then the equation becomes

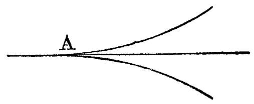
$$y^2 = (x-a)^2 (x-b),$$



where a is less than b; the oval has shrunk into a point (A), and the curve is of the annexed form.



This example sufficiently shows the analogy between conjugate points, and double points the tangents at which are real. If we suppose $a = b = c$, the equation becomes $y^2 = (x-a)^3$, the point A becomes a cusp, as in II. of last Article, and the tangent at the cusp meets the curve in three coincident points A, B, C.



33. If in the general equation A, B, C, D, E, F were all $= 0$, then the origin would be a triple point, every line through the origin meeting the curve in three coincident points; and it is easy to see, as before, that at a triple point there are three tangents, which are the three lines represented by the equation $u_3 = 0$.

We may also, as before, distinguish four species of triple points, according as the three tangents are (1) real and distinct, (2) one real and two coincident, (3) all three coincident, or (4) one real, and two imaginary. The last kind of triple point is worth notice, as to the eye it does not appear to differ from any other point on the curve.

We may, in like manner, investigate the conditions that the origin should be a multiple point of any higher degree (k). The coefficients of all terms of a degree below k will vanish, and the equation will be of the form

$$u_k + u_{k+1} + \&c. = 0.$$

At the multiple point there can be drawn k tangents, represented by the equation $u_k = 0$; and the nature of the multiple point varies according as the roots of this equation are all real and unequal, or two or more of them equal or imaginary.

34. Before quitting the subject of multiple points, we shall mention some simple considerations which fix a limit to the num-

ber of such points which a curve of the n^{th} degree can possess, when it does not break up into others of lower dimensions.

For example, a curve of the third degree cannot have two double points; for if it had, the line joining them must be considered as meeting the curve in four points; but more than three points of a curve of the third degree cannot lie on a right line, unless the curve consist of this right line and a conic.

Again, a curve of the fourth degree cannot have four double points; for if it had, the conic determined by these and any fifth point of the curve must be considered as meeting the curve in nine* points; whereas no conic, distinct from the curve, can meet it in more than 2×4 points. And, in general, a curve of the n^{th} degree cannot have more than $\frac{(n-1)(n-2)}{2}$ double points; for if it had one more, through these $\frac{(n-1)(n-2)}{2}+1$ and $n-3$ other points of the curve, we could describe a curve of the degree $n-2$ (Art. 20), which must be considered as meeting the given curve in $2\left\{\frac{(n-1)(n-2)}{2}+1\right\}+n-3$ points, or in $n(n-2)+1$ points, which is impossible if the given curve be a proper curve.

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* If a point of intersection of two curves be a double point on one of them, that intersection must be reckoned as two, and the curves can only intersect in $n-2$ other points. If it be a double point on both, the intersection must be reckoned as four. And in general if it be on the one curve a multiple point of the degree k, and on the other of the degree l, that intersection must be counted as kl. Thus, for example, a system of k right lines meets a system of l right lines in kl points; but if all the lines of the first system pass through a point on a line of the second system, that point clearly counts as k intersections, and the lines intersect only in $k(l-1)$ other points. And if every line of both systems pass through the same point, that point counts as kl intersections, and the lines meet nowhere else.

If two curves touch at their point of intersection, the point of contact will, of course, count as two intersections, since they have two coincident points common. If the point of intersection be a multiple point on one or both curves, and if one of the tangents at the multiple point were common to both curves, we should add one to the number of intersections to which it has been already shown that the multiple point was equivalent; for, besides the points just proved to be common, they have a consecutive point common on one of the branches through the multiple point.

The reader will have no difficulty in seeing the effect of any combination of tangents and multiple points.

35. So in like manner we can fix limits to the number of higher multiple points which can exist on a given curve. Thus, for instance, if a curve of the n^{th} degree have a multiple point of the degree $n-1$, it can have no other multiple point; for if it had, the line joining the two would meet the curve in more than n points. If it have a multiple point of the degree $n-2$, it can have no other higher than a double point, and of these it can be proved, as in the last Article, it can have no more than $\frac{(n-2)(n-3)}{2}$. There is no difficulty in applying the same principles to any combination of multiple points of different degrees, but it is not easy to include the results in a general formula. Of course the demonstration given only shows that curves cannot have more than a certain number of multiple points, but does not make it certain that they can always have so many.*

36. We can show that, when a curve has its maximum number of multiple points, there will be in general certain relations connecting them, so that, a certain number of the multiple points being given, the position of the rest is determined.

To show this we must inquire to how many conditions being given a multiple point is equivalent? If we were given a double point on a curve we might take it for the origin, and we have seen (Art. 30) that three terms of the equation will then vanish. The constants at our disposal are then three less than in the general case: the double point is therefore equivalent to three conditions.

* The principles employed in this and the preceding article have been laid down in Cramer's Introduction à l'Analyse des Courbes. The results thus obtained were some time ago assailed by a writer (M. Coste) in Liouville's Journal (tom. vii. p. 184), who, without attempting to controvert the arguments used by Cramer, endeavours practically to confute him by producing examples of curves having more double points than are consistent with the preceding theory. Thus the curve represented by the equation

$$16x^4 + y^4 - 8x^2 - 2y^2 + 1 = 0,$$

is produced as an example of a curve of the fourth degree, having four double points, viz., the points $(y = 0,\ 4x^2 = 1)$, $(x = 0,\ y^2 = 1)$; and it is stated to consist of two distinct ovals which intersect in these points. The ovals, however, are no other than the two ellipses,

$$4x^2 \pm 2\sqrt{2}\,xy + y^2 - 1 = 0;$$

and the theory given above only determines three as the maximum number of double points on a *proper* curve of the fourth degree.

Were we given also the tangents at the double point, this would be equivalent to two conditions more, for, in addition to $A=0$, $B=0$, $C=0$, we should be given the quantities $\frac{E}{D}, \frac{F}{D}$. Being given a triple point is equivalent to six conditions; for, making it the origin, the six lowest terms of the equation vanish: and so, in general, being given a multiple point of the degree k is equivalent to $\frac{k(k+1)}{2}$ conditions.

We can now see that any three points taken arbitrarily may be double points on a curve of the fourth degree; for the three are equivalent to but nine conditions. But the tangents at all these double points cannot also be assumed arbitrarily; for being given the three double points and these three pairs of tangents is equivalent to fifteen conditions, one more than enough to determine the curve. There must then be some relation connecting these tangents; and, in fact, we shall prove afterwards that these six tangents all touch the same conic section, so that, given five, the sixth is determined.

Twenty conditions determine a curve of the fifth degree. We may then assume arbitrarily its six double points, and also the pair of tangents at any one of them; but the curve is then completely determined, and therefore also the pairs of tangents at the other five.

Twenty-seven conditions determine a curve of the sixth degree. If then we wish to describe such a curve having ten double points, we cannot even assume so many as nine of them arbitrarily. For these nine would completely determine the curve, which then might not happen to possess any other double point: there must then be some relations connecting the ten double points.

And so in like manner for curves of higher degrees, when they have their maximum number of double points there must be a still greater number of relations connecting them. Except in the case of curves of the fourth degree, we are not aware that any attempt has been made to express these relations geometrically, but there must remain an extensive class of theorems of this nature still to be discovered.

37. What has been said is sufficient to enable the reader to form a conception of the nature of multiple points on curves. We shall now proceed to show that a curve may in like manner have multiple tangents; or, in other words, that there may be lines which touch the curve in two or more points. What are commonly called the "singular points" of curves may be reduced to the two classes, either of multiple points, or of points of contact of multiple tangents. As we introduced multiple points to the reader by an examination of the particular case where the origin was a multiple point, so it will be more simple to commence our discussion of multiple tangents by examining the condition that the axis ($y = 0$) should be a multiple tangent.

We find in general the points where this line meets the curve by making $y = 0$ in the general equation, whence we get

$$A + Bx + Dx^2 + Gx^3 + \ldots . Px^n = 0,$$

an equation which can be reduced to the form

$$P(x - a)(x - b)(x - c)(x - d) \text{ \&c.} = 0,$$

where a, b, &c., are the values of x for the points where the axis meets the curve.

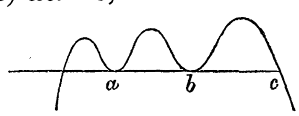
The axis will be a tangent when two of these points coincide, that is, when the equation can be reduced to the form

$$P(x - a)^2 (x - b) \text{ \&c.} = 0.$$

The axis then touches the curve at the point $y = 0$, $x = a$. If $A = 0$, $B = 0$, the axis touches the curve at the origin.

I. But now if the equation have two distinct pairs of equal roots, that is to say, if it be of the form

$$P(x - a)^2 (x - b)^2 (x - c) \text{ \&c.} = 0,$$



the axis is a double tangent at the two real points, $x = a$, $x = b$. It is evident that such a tangent, meeting the curve in two pairs of coincident points, cannot occur in any curve of a degree lower than the fourth.

II. But, secondly, the axis must be considered equally as a double tangent, even if the two points of contact were imaginary, that is to say, if the equation were still of the form

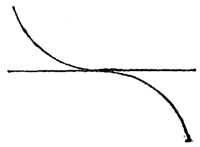
$$P(x^2 + px + q)^2 (x - c) \text{ \&c.} = 0,$$

even though x^2+px+q could not be resolved into two real factors, $(x-a)$, $(x-b)$.

III. Thirdly, the equation may be of the form

$$P(x-a)^3 (x-c) \text{ \&c.} = 0.$$

The axis then meets the curve in three consecutive points. In general, taking three consecutive points on a curve, the line joining the first and second of these is one tangent, and the line joining the second and third is the consecutive tangent. In the present case, therefore, two consecutive tangents coincide. Hence too, in such a case, the axis may be called a *stationary* tangent; for if we consider the curve as the *envelope* of a moveable line, in this case two consecutive positions of the moveable line coincide. The point of contact of a stationary tangent is called *a point of inflexion.*



If $A=0$, $B=0$, $D=0$, the origin is a point of inflexion, and $y=0$ the tangent at it; since then the equation is of the form

$$Px^3 (x-c) \text{ \&c.} = 0.$$

38. The reader will have observed the strict correspondence between our division of double tangents and our division of double points (Art. 31); that, as every double point has two tangents, so every double tangent has two points of contact; and that the two tangents in the first case, and the two points in the second, may be either real and distinct, real and coincident, or imaginary. The learner, on finding that the tangent at a point of inflexion is a double tangent whose points of contact coincide, is likely to imagine that it must meet the curve in four, not three, coincident points; since, if (see figure, p. 34) the points a, b coincide, the line joining them will still meet the curve in four points. But a line which meets the curve in four coincident points is a *triple*, not a *double* tangent; for it is a tangent in virtue of joining the first of the coincident points to the second, the second to the third, and the third to the fourth. Just as a common chord meeting the curve in two points, a, b, becomes a tangent when the points a, b coincide; so a double tangent touching the curve at a and b would become a triple tangent when a and b coincide; and the only way to obtain a double tangent with coincident points of contact is to

take a line touching the curve at a, and cutting it at b, and then to imagine the points a and b to coincide.

39. We wish next to show that whereas in ordinary cases the curve lies altogether at the same side of the tangent, at a point of inflexion the curve crosses the tangent, and lies part on one side, and part on the other.

This is a particular case of the following more general theorem: *Two curves which have common an even number of consecutive points touch without cutting; those which have common an odd number of consecutive points cross one another at their point of meeting.*

Let the equations of the two curves be $y = \phi x$, $y = \psi x$; let them intersect at the point $x = a$; then, by Taylor's theorem, the values of the ordinates of the two curves, for the point $x = a + h$, are

$$y_{,} = \phi + \frac{d\phi}{dx}\frac{h}{1} + \frac{d^2\phi}{dx^2}\frac{h^2}{1.2} + \frac{d^3\phi}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

$$y_{,,} = \psi + \frac{d\psi}{dx}\frac{h}{1} + \frac{d^2\psi}{dx^2}\frac{h^2}{1.2} + \frac{d^3\psi}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

where $\phi, \psi, \frac{d\phi}{dx}$, &c., are the values of $\phi x, \psi x, \frac{d\phi x}{dx}$, &c., when $x = a$. Now, by hypothesis, $\phi = \psi$, since the curves intersect at the point $x = a$, therefore

$$y_{,} - y_{,,} = \left(\frac{d\phi}{dx} - \frac{d\psi}{dx}\right)\frac{h}{1} + \left(\frac{d^2\phi}{dx^2} - \frac{d^2\psi}{dx^2}\right)\frac{h^2}{1.2} + \left(\frac{d^3\phi}{dx^3} - \frac{d^3\psi}{dx^3}\right)\frac{h^3}{1.2.3} + \&c.$$

Now, by the principles of the differential calculus, when h is indefinitely small, the sign of the sum of this series is the same as the sign of its first term, but the sign of this term is changed when the sign of h is changed; therefore if, at the infinitely near point $(x = a + h)$, the ordinate of the curve ϕ be greater than that of the curve ψ, it will be less at the point $(x = a - h)$. Hence if two curves have one point common, in general, that which is uppermost at one side of the point will be undermost at the other.

But now suppose that $\frac{d\phi}{dx} = \frac{d\psi}{dx}$, the first term of the series will then be $\left(\frac{d^2\phi}{dx^2} - \frac{d^2\psi}{dx^2}\right)\frac{h^2}{1.2}$, which does *not* change sign when h changes sign. The same curve, therefore, which is uppermost on

one side of the given point, will be uppermost also on the other. But when $\frac{d\phi}{dx} = \frac{d\psi}{dx}$, the curves are manifestly closer to each other than in the previous case, since the difference of the ordinates no longer involves the first power of h; which is equivalent to what is expressed geometrically, by saying that the curves have two consecutive points common. Or the same thing may be shown thus: $x'y'$, $x''y''$ being the co-ordinates to rectangular axes of any two points on a curve, $\frac{y'-y''}{x'-x''}$ is plainly the tangent of the angle which the chord joining them makes with the axis of x: but if the points coincide, we learn that *the value of* $\frac{dy}{dx}$ *for the given point expresses the tangent of the angle which the line joining it to the consecutive point* (*i. e. the tangent*) *makes with the axis of* x; consequently, if two curves have a point common, and $\frac{dy}{dx}$ for that point the same for both curves, it follows that the consecutive point is also common.

40. When the curves have three consecutive points common, we shall have $\frac{d^2\phi}{dx^2} = \frac{d^2\psi}{dx^2}$; the first term of the series for $y_, - y_{,,}$ is $\left(\frac{d^3\phi}{dx^3} - \frac{d^3\psi}{dx^3}\right)\frac{h^3}{1.2.3}$, which does change its sign with h, and therefore, as before, the curves cross at the given point. And so, in general, if the expansion of $y_, - y_{,,}$ commence with an even power of h, it will not change sign with h, and therefore the curves touch without crossing; but if it commence with an odd power of h, the sign will change with h, and therefore the curves cross at the given point.

The reader has already had an illustration of this, in the case of the circle which osculates a conic at any point, and which, in general, having three points common with the curve, touches and crosses the curve (*Conics*, p. 205); but at the extremities of the axes the osculating circle passes through four consecutive points, and touches without crossing.

The same investigation applies when one of the curves become a right line. A tangent, therefore, at a point of inflexion, or any

line meeting the curve in an odd number of consecutive points, is crossed by the curve: but a tangent which meets the curve in an even number of consecutive points has the neighbouring part of the curve all at the same side of it.

41. The axis $y = 0$ will be a triple tangent when the equation which determines the points where it meets the curve is of the form
$$P(x-a)^2(x-b)^2(x-c)^2(x-d)\ \&c. = 0.$$
It is evident such a tangent cannot occur in a curve of any degree lower than the sixth. We may, as in Art. 33, distinguish four species of triple tangents according as the points of contact are real and distinct, one real and two imaginary, one real and two coincident, or all three coincident. The last will be the case when the equation is of the form
$$P(x-a)^4(x-b)\ \&c. = 0;$$
and the axis meets the curve in four coincident points, and is therefore a triple tangent, as was shown in Art. 38. The point of contact of such a tangent is called *a point of undulation.* In like manner there may be multiple tangents of still higher orders, or again, points of undulation of higher orders, arising when a line meets the curve in more than four coincident points. Cramer calls those points at which the tangent meets the curve in an odd number of consecutive points, points of *visible inflexion*, to distinguish them from those *points de serpentement*, or points of undulation, which do not, to the eye, differ from ordinary points on the curve.

42. We have hitherto only illustrated the case where the origin is a multiple point, or one of the axes a multiple tangent; it is evident, however, that the form of the equation might, in like manner, betray the existence of multiple points and tangents situated anywhere.

I. For instance, if the equation be of the form
$$\alpha\phi + \beta\psi = 0,$$
where α, β are the equations of any two right lines, and ϕ, ψ are any functions of the co-ordinates, then $\alpha\beta$ is one point on the curve. The equation of the tangent at this point is
$$\alpha\phi' + \beta\psi' = 0,$$

where ϕ', ψ' are the forms which ϕ and ψ assume when we introduce the conditions $\alpha = 0$, $\beta = 0$. For if we seek the $n-1$ points, in which any line through $\alpha\beta$, $(\alpha - k\beta)$ meets the curve, we get an equation of the form

$$\beta\,\{k(\phi' + M\beta + N\beta^2 + \&c.) + (\psi' + M'\beta + N'\beta^2 + \&c.)\} = 0;$$

and in order that a second root of this should be $\beta = 0$, we must have $k\phi' + \psi' = 0$; whence, substituting for k, $\frac{\alpha}{\beta}$, we get for the equation of the tangent,

$$\alpha\phi' + \beta\psi' = 0.$$

II. In general the curve represented by

$$\alpha\beta\gamma\delta \;\&c. = \alpha_{,}\beta_{,}\gamma_{,}\delta_{,} \;\&c.$$

passes through the points

$$\alpha\alpha_{,},\ \alpha\beta_{,},\ \alpha\gamma_{,},\ \&c.,\ \beta\beta_{,},\ \beta\gamma_{,},\ \&c.,\ \gamma\gamma_{,},\ \&c.$$

III. If the equation be of the form

$$\alpha\phi + \beta^2\psi = 0,$$

we see (as at *Conics*, p. 214), that α is the tangent at the point $\alpha\beta$, for two of the points in which this line meets the curve coincide.

Or again, if the curve be

$$t_1\, t_2\, t_3 \,\ldots\, t_n + \beta^2\phi = 0,$$

t_1, &c., are the tangents at the n points, where β meets the curve.

The form of the equation shows that *if the points of contact of n tangents lie on a right line β, the remaining points where these tangents meet the curve lie on a curve of the $n - 2^{nd}$ degree ϕ.*

IV. If the equation be of the form

$$\alpha^2\phi + \alpha\beta\psi + \beta^2\chi = 0.$$

If we seek the points where any line $(\alpha = k\beta)$ through $\alpha\beta$ meets the curve, we find that two of these always coincide with $\alpha\beta$, and therefore that this is a double point. It appears precisely as in I., and in Art. 30, that the tangents at this double point are

$$\alpha^2\phi' + \alpha\beta\psi' + \beta^2\chi' = 0,$$

where ϕ', ψ', χ' are the values which these functions take for the co-ordinates of the point $\alpha = 0$, $\beta = 0$.

V. So again, if the equation be of the form

$$\alpha^3\phi + \alpha^2\beta\psi + \alpha\beta^2\chi + \beta^3\omega = 0,$$

the point $\alpha\beta$ is a triple point; the three tangents being given by the equation

$$\alpha^3\phi' + \alpha^2\beta\psi' + \alpha\beta^2\chi' + \beta^3\omega' = 0.$$

VI. If the equation be of the form

$$\alpha\phi + \beta^2\gamma^2\psi = 0,$$

α is a double tangent at the points $\alpha\beta$, $\alpha\gamma$.

VII. If the equation be of the form

$$\alpha\phi + \beta^3\psi = 0,$$

$\alpha\beta$ is a point of inflexion, and α the tangent at it.

43. We shall first illustrate the last Article, by showing how the equation enables us to discern the nature of the points of the curve at an infinite distance. The trilinear equation is (Art. 19)

$$u_n + u_{n-1}z + u_{n-2}z^2 + \&c. = 0.$$

The directions of the n points at infinity are found (by making $z = 0$ in the equation) from the equation $u_n = 0$, which solved for $y:x$, is of the form

$$(y - m_1x)(y - m_2x)(y - m_3x)(\&c.)(y - m_nx) = 0.$$

A curve of the n^{th} degree has, in general, n asymptotes, namely, the tangents at the n points, where z, the line at infinity, meets the curve. We can find their equations readily as follows, when the equation $u_n = 0$ has been solved for $y:x$. It appears from III. of the last Article that if the equation were reduced to the form

$$t_1 t_2 \ldots t_n + z^2\phi = 0,$$

t_1, &c., would be the n asymptotes. But the given equation

$$(y - m_1x)(y - m_2x)\,\&c. + zu_{n-1} + z^2u_{n-2} + \&c. = 0$$

may always be reduced to the form

$$(y - m_1x + \lambda_1)(y - m_2x + \lambda_2)\,\&c. = z^2\phi;$$

for the terms of the n^{th} degree in x and y are obviously the same for both equations, and the n arbitraries, λ_1, &c., in the second, can be so determined as to make the n terms of the $n - 1^{st}$ degree the same for both equations.

The reader will have no difficulty in understanding this method, if he tries to apply it to a particular example; for instance,

$$(x + y)(2x + y)(3x + y) + 17x^2 + 11xy + 2y^2 + 12x + 10y + 36 = 0,$$

which it is desired to throw into the form

$$(x + y + \lambda_1)(2x + y + \lambda_2)(3x + y + \lambda_3) + Ax + By + C = 0.$$

To determine $\lambda_1, \lambda_2, \lambda_3$ we should then have the three equations

$$6\lambda_1 + 3\lambda_2 + 2\lambda_3 = 17,\quad 5\lambda_1 + 4\lambda_2 + 3\lambda_3 = 11,\quad \lambda_1 + \lambda_2 + \lambda_3 = 2;$$

and the equation may be reduced to the form

$$(x + y + 4)(2x + y - 3)(3x + y + 1) + 43x + 21y + 48 = 0.$$

44. If two roots of the equation $u_n = 0$ be equal ($m_1 = m_2$), the general equation takes the form $(y - m_1x)^2 \phi + z\psi = 0$; two of the points where z meets the curve coincide, and the line at infinity is therefore a tangent to the curve.

Should three roots of this equation be equal, the line at infinity meets the curve in three coincident points, and therefore touches at a point of inflexion.

If in the general equation the coefficient of $y^n = 0$, the axis of y would pass through a point at infinity, and we have evidently only an equation of the $n - 1^{st}$ degree to determine the remaining points where it meets the curve.

Should the coefficient of y^{n-1} also vanish, the axis of y will be an asymptote.

If two factors of u_n be equal, and one of them also a factor in u_{n-1}, then the curve has a double point at infinity; for the equation is of the form

$$(y - m_1x)^2 \phi + z(y - m_1x)\psi + z^2\chi = 0.$$

45. We shall in a future section show how the singular points of a curve may, in general, be found. But the application of the general methods being usually a work of some difficulty, the examples given in works on the differential calculus are, for the most part, cases where the existence of the singular points more readily appears from mere inspection of the equations; a selection, including all the most difficult of these examples, will therefore serve to illustrate the preceding Articles. (See Gregory's Examples, p. 170, &c.)

(1.) $x^4 - ayx^2 + by^3 = 0.$ (2.) $x^4 - 2ax^2y + 2x^2y^2 + ay^3 + y^4 = 0.$

In both cases the origin is a triple point. The tangents of the first are given by the equation $ax^2y = by^3$; and of the second by the

equation $2x^2y = y^3$. By Art. 34, neither curve can have any other multiple point.

(3.) $ay^2 - x^3 \pm bx^2 = 0$.

The origin is a double point, whose tangents are given by the equation $ay^2 \pm bx^2 = 0$.

If the sign be given positive, the origin is a conjugate point.

(4.) $(x^2 - a^2)^2 = ay^2(2y + 3a)$, or $(x - a)^2 (x + a)^2 = ay^2 (2y + 3a)$.

Here evidently $(x - a, y)$ and $(x + a, y)$ are double points. To get the tangents at the first, we are to make $x = a$, $y = 0$ in the parts which multiply $(x - a)^2$, y^2, and we get

$$4(x - a)^2 = 3y^2.$$

In like manner for the tangents at the other double point,

$$4(x + a)^2 = 3y^2.$$

The curve has a third double point, whose existence can be shown by throwing the equation into the form

$$x^2(x^2 - 2a^2) = a(2y - a)(y + a)^2.$$

Hence $(x, y + a)$ is a double point, and the tangents at it are

$$2(x^2) = 3(y + a)^2.$$

Having found these three, we know, by Art. 34, that the curve can have no other multiple point.

(5.) $(by - cx)^2 = (x - a)^5$.

The point $(by - cx, x - a)$ is a cusp of such a nature that the tangent at it meets the curve in five consecutive points.

(6.) $x^4(x + b) = a^3y^2$.

The origin is a double point, the tangent at which meets the curve in four consecutive points. There is a triple point at infinity, to which the line at infinity is the only tangent. The line $x + b$ touches the curve where it meets the axis of x, and also at a point of inflexion at infinity.

(7.) $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = 0$.

This equation cleared of radicals becomes

$$(x^2 + y^2 + z^2)^3 = 27x^2y^2z^2;$$

and in this form the existence of six cusps is manifest, for each of the points where x meet $y^2 + z^2$ is a double point, and x the only tangent at it. Similarly for $(y, x^2 + z^2)$ and $(z, x^2 + y^2)$.

The curve has also four other double points, viz., $(x \pm y, x \pm z)$.

This can be proved by putting $y \mp x = u$, $z \mp x = v$; and therefore

$$y = u \pm x,\ \ z = v \pm x.$$

Substituting these values in the given equation, it is of the form

$$u^2\phi + uv\psi + v^2\chi.$$

The tangents at any of the double points will be found to be given by the equation

$$u^2 \pm uv + v^2 = 0,$$

and therefore the double points in question are conjugate points.

SECT. III.—MISCELLANEOUS PROPERTIES COMMON TO ALL ALGEBRAIC CURVES.

46. The theorem given (*Conics*, p. 136) may be generalized as follows: *If through any point* O *two chords be drawn, meeting a curve of the* n^{th} *degree in the points* $R_1 R_2 \ldots R_n$, $S_1 S_2 \ldots S_n$, *then the ratio of the products* $\dfrac{OR_1 . OR_2 \ldots OR_n}{OS_1 . OS_2 \ldots OS_n}$ *will be constant, whatever be the position of the point* O, *provided that the directions of the lines* OR, OS *be constant.**

And the proof is the same as that already given in the case of conic sections. From the polar equation of the curve, Art. 28, we see that the product of all the values of the radius vector on a line through the origin making an angle θ with the axis of x, is

$$= \frac{A}{P \cos^n\theta + Q \cos^{n-1}\theta \sin\theta + \&c.},$$

and the same product for any other line is

$$= \frac{A}{P \cos^n\theta_, + Q \cos^n\theta_, \sin\theta_, + \&c.}.$$

The ratio is therefore

$$\frac{P \cos^n\theta + Q \cos^{n-1}\theta \sin\theta + \&c.}{P \cos^n\theta_, + Q \cos^{n-1}\theta_, \sin\theta_, + \&c.}.$$

But we have seen (*Conics*, p. 122) that by a transformation to any parallel axes the coefficients of the highest powers of the variables, and therefore this ratio, will be unaltered.

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* This theorem was first given by Newton, in his Enumeratio Linearum Tertii Ordinis.

We may (as at *Conics*, p. 136) express the same theorem thus: *If through two fixed points,* O *and* o, *any two parallel lines be drawn, then the ratio of the products* $OR_1 . OR_2 . OR_3 ..$ *&c.* : $or_1 . or_2 . or_3$, *&c. will be constant, whatever be the common direction of these lines.*

For the value of the second product is $\dfrac{A'}{P\cos^n\theta + \&c.}$, where A′ is the absolute term when o is made the origin; and the ratio of the products is A : A′, and independent of θ. We have seen (*Conics*, p. 122) that the new absolute term will be the result of substituting the co-ordinates of o in the given equation. We see, therefore, that the result of such a substitution is always proportional to the product of the segments intercepted between o and the curve on a line whose direction is given (*Conics*, p. 217).

47. From the preceding theorem is deduced at once Carnot's theorem, of which we have given a particular case (*Conics*, p. 263). Let each of the sides of a polygon A, B, C, &c., meet a curve of the n^{th} degree in n real points. We shall denote by (B)′ the continued product of the n segments made on the side BC between B and the curve: by ′(B) the product of the segments made on the side BA. Then

$$(A)'\,(B)'\,(C)'\,(D)'\,\&c. = {}'(A)\,{}'(B)\,{}'(C)\,{}'(D)\,\&c.$$

For through any point draw radii vectores parallel to the sides of the polygon, and denote the continued product of the segments on each of these lines by (a) (b) (c), &c., then

$$\begin{gathered} {}'(B) : (B)' :: (a) : (b) \\ {}'(C) : (C)' :: (b) : (c) \\ {}'(D) : (D)' :: (c) : (d) \\ \&c. \end{gathered}$$

and compounding all these ratios, the truth of the theorem is evident.

48. Some ambiguity will be avoided by attention to the signs $\pm$. The reader will remember (see *Conics*, p. 6) that we have distinguished the cases where a line is cut internally and externally in the same ratio, by using the sign + in the first case, and − in the second. If then we write the formula

$${}'(A)\,{}'(B)\,{}'(C)\,\&c. = \pm\,(A)'\,(B)'\,(C)'\,\&c.,$$

and inquire which sign is to be used; it is easy to see, by taking the case of a very small polygon, that the formula must be true when *every* side of the polygon is cut externally, and, therefore, when every term in the product $(A)'$ has a sign opposite to that of the corresponding term in the product $'(B)$. If the curve be of an even degree the products $(A)'$ and $'(B)$ will have the same sign; if it be of an odd degree, they will have opposite signs. But even if the curve be of an odd degree, still if the number of sides of the polygon be even, the number of products will be even, and therefore the two continued products $(A)'\,(B)'\,(C)'$ &c., $'(A)\,'(B)\,'(C)$ &c., will have the same sign. We learn then that in the preceding equation the sign + is to be used when either the degree of the curve or the number of sides of the polygon is even, but that when both are odd the sign − must be used.

49. We shall give a few examples to illustrate the fertility of this theorem. (See Plücker's System der Analytischen Geometrie, p. 44.)

(1.) Let a right line meet the sides of a triangle AB, BC, CA, in the points c, a, b. Then

$$Ac \,.\, Ba \,.\, Cb = -\,Ab \,.\, Bc \,.\, Ca, \quad (\textit{Conics}, \text{ p. } 33),$$

and the sign shows that, if it cut two sides internally, it must cut the third externally. The equation

$$Ac_{,} \,.\, Ba \,.\, Cb = +\,Ab \,.\, Bc_{,} \,.\, Ca, \quad (\textit{Conics}, \text{ p. } 34),$$

will be fulfilled if the three lines Aa, Bb, $Cc_{,}$, meet in a point; and the line AB is cut harmonically in the points c and $c_{,}$.

(2.) Let us now pass to curves of the second degree, and suppose each side of the triangle to touch the curve in the points a, b, c. Carnot's theorem gives us

$$Ac^2 .\, Ba^2 .\, Cb^2 = +\,Ab^2 .\, Bc^2 .\, Ca^2;$$

and, therefore, $$Ac \,.\, Ba \,.\, Cb = \pm\,Ab \,.\, Bc \,.\, Ca.$$

The lower sign cannot be used, since no line can meet a conic in three points: we learn then that if a conic be inscribed in a triangle, the lines joining each vertex to the opposite point of contact meet in a point.

(3.) Again, let a, b, c be points of inflexion on a curve of the

third degree, at which BC, CA, AB are tangents; then, by Carnot's theorem,

$$Ac^3 \,.\, Ba^3 \,.\, Cb^3 = - Ab^3 \,.\, Bc^3 \,.\, Ca^3,$$

the only real root of which is

$$Ac \,.\, Ba \,.\, Cb = - Ab \,.\, Bc \,.\, Ca.$$

Hence, *if a curve of the third degree have three real points of inflexion, they must lie on one right line.* Hence too a curve of the third degree can have only three real points of inflexion; for this Article would show that *all* the real points of inflexion must lie on a right line; and a right line can only meet the curve in three points.

The same reasoning proves that if any curve of an odd degree have three real points, at each of which the tangent meets the curve in n points, these three points must lie on one right line.

(4.) Again, let a curve of the fourth degree have three double tangents; we have

$$Ac^2 . Ac_,^2 . Ba^2 . Ba_,^2 . Cb^2 . Cb_,^2 = Ab^2 . Ab_,^2 . Bc^2 . Bc_,^2 . Ca^2 . Ca_,^2,$$

whence

$$Ac \,.\, Ac_, \,.\, Ba \,.\, Ba_, \,.\, Cb \,.\, Cb_, = \pm Ab \,.\, Ab_, \,.\, Bc \,.\, Bc_, \,.\, Ca \,.\, Ca_, \,;$$

but on account of the double sign we can only infer that "if a curve of the fourth degree have three double tangents, the conic through five of the points of contact will either pass through the sixth, or through the point which, with the sixth, divides harmonically the side of the triangle on which the sixth lies."

50. There are some particular cases for which Carnot's theorem requires to be modified. First, if one of the angles (A) of the polygon were at infinity, that is to say, if two adjacent sides be parallel, then $(A)'$ ultimately $= '(A)$, and we still have the equation

$$(B)'\,(C)'\,\&c. = '(B)\,'(C)\,\&c.$$

Secondly, if one of the angles (A) were on the curve; then one of the n terms vanishes in each of the products $(A)'$ and $'(A)$; but now, since the ratio of any two lines $\dfrac{AR}{AR'} = \dfrac{\sin RR'A}{\sin R'RA}$, we may substitute for the ratio of these two vanishing sides, the ratio of the sines of the angles which the sides of the polygon at A make with the tangent at A, and the theorem becomes

$$\frac{(A)'\,(B)'\,(C)'\ \&c.}{\sin \alpha} = \frac{'(A)\ '(B)\ '(C)\ \&c.}{\sin \alpha'},$$

where $(A)'\ '(A)$ have each but $n-1$ factors, and where α, α' are the angles which the sides on which $(A)', '(A)$ are measured make with the tangent at A. In this manner we can deduce that, "if any polygon be inscribed in a conic, the continued product of the sines of the angles which each side makes with the tangent at its right hand extremity is equal to the similar product of the sines of the angles made with the tangent at the other extremity."

DIAMETERS.

51. If there be n points in a right line, a point on the line such that the algebraic sum of its distances from these points shall vanish, is called the *centre of mean distances* of the given points. Let the distance of the centre from any assumed point on the line be y, let that of the other points be y_1, y_2, y_3, &c., then the distances of the centre from the given points are $y - y_1$, $y - y_2$, &c., and the condition given by the definition is

$$\Sigma(y - y_1) = 0, \text{ or } ny - \Sigma(y_1) = 0;$$

whence we learn that the distance of any assumed point from the centre is equal to the sum of the distances of the assumed point from the given points, divided by the number of these points; or is equal to the *mean distance* of the assumed point from the given points. Thus if there be only two given points, the centre of mean distances is the middle point of the line joining them, and the distance of any point on the line from the middle point is half the sum of its distances from the two given points.

The well-known properties of the diameters of conics have been generalized by Newton into the following theorem, true for all algebraic curves: *If on each of a system of parallel chords of a curve of the n^{th} degree there be taken the centre of mean distances of the n points, where the chord meets the curve, the locus of this centre is a right line, which may be called the* diameter *corresponding to the given system of parallel chords.*

To prove this theorem, we adopt the same method of investigation as in the case of conic sections. (*Conics*, p. 129.) The origin would be the centre of mean distances for a chord making

an angle θ with the axis of x, if, when we transform to polar co-ordinates by substituting $\rho \cos\theta$, $\rho \sin\theta$ (or, in case of oblique axes, $m\rho$, $n\rho$), for x and y, θ be such as to cause the coefficient of ρ^{n-1} to vanish. If we seek then the condition that any other point $x'y'$ should be the centre of mean distances for a parallel chord, we must examine what relation should exist between $x'y'$, in order that when we transform the axes to this point the new coefficient of ρ^{n-1} should vanish for the same value of θ. But when the given equation $U = 0$ is transformed to parallel axes by substituting $x + x'$, $y + y'$, for x and y, it becomes (Lacroix's Differential Calculus, p. 54)

$$U + x'\frac{dU}{dx} + y'\frac{dU}{dy} + \tfrac{1}{2}\left(x'^2\frac{d^2U}{dx^2} + 2x'y'\frac{d^2U}{dxdy} + y'^2\frac{d^2U}{dy^2}\right) + \&c. = 0\,;$$

only the three first terms can contain powers of the variables as high as the $(n-1)^{st}$, and since these involve $x'y'$ only in the first degree, the required locus must be a right line. Its equation is, in fact,

$$x\frac{du_n}{dx} + y\frac{du_n}{dy} + u_{n-1} = 0,$$

where in u_n, u_{n-1}, $\cos\theta$ and $\sin\theta$ (or, if the axes be oblique, m and n) have been substituted for x and y.

52. Newton has also remarked that if any chord cut the curve and its asymptotes, the same point will be the centre of mean distances for both, and that therefore the algebraic sum of the intercepts between the curve and its asymptotes $= 0$. This is the extension of the well-known theorem (*Conics*, p. 172). The truth of it follows at once from the equation of a diameter given in the last Article, and from what was proved (Art. 43) that the terms u_n, u_{n-1}, are the same in the equation of the curve and in that of its n asymptotes.

53. We may in like manner seek the locus of a point such that the sum *of the products in pairs* of the intercepts, measured in a given direction between it and the curve, should vanish. The origin would be such a point if the coefficient of ρ^{n-2} vanished for the given value of θ, and the locus is found, as in Art. 51, by examining what relation should exist between x' and y' that the coefficient of ρ^{n-2} in the transformed equation should vanish. But

since the terms of the $(n-2)^{nd}$ degree in x and y involve no powers higher than the second of x' and y', the locus will be a conic section, which we shall call the *diametral conic.*

Its equation is readily seen to be

$$u_{n-2} + x\frac{du_{n-1}}{dx} + y\frac{du_{n-1}}{dy} + \tfrac{1}{2}\left(x^2\frac{d^2u_n}{dx^2} + 2xy\frac{d^2u_n}{dxdy} + y^2\frac{d^2u_n}{dy^2}\right) = 0,$$

where, in u_{n-2}, &c., $\cos\theta$ and $\sin\theta$ have been substituted for x and y. The distance of any point from either point on the diametral conic being y, and from the curve y_1, y_2, &c., we have by the definition

$$\Sigma\,(y-y_1)\,(y-y_2) = 0.$$

The number of terms in this sum is the same as the number of combinations in pairs of n things, and is therefore $= \dfrac{n\,(n-1)}{2}$. This, therefore, will be the coefficient of y^2 when we multiply out each of these products, and add them together. In the same case the coefficient of y will consist of $\dfrac{n\,(n-1)}{2}$ terms, each of the form $-(y_1+y_2)$, and since it must involve the n quantities y_1, y_2, &c., symmetrically, it must be $-(n-1)\,\Sigma y$. Hence

$$\Sigma\,(y-y_1)\,(y-y_2) = \frac{n\,(n-1)}{2}\,y^2 - (n-1)\,y\,\Sigma(y_1) + \Sigma(y_1y_2) = 0.$$

This quadratic gives the distances of any point from the diametral conic when we know its distances from the curve. $\dfrac{n(n-1)}{2}$ times the product of these two distances $= \Sigma(y_1y_2)$, or *the product of the distances from the diametral conic is equal to the mean product in pairs of the distances from the curve*, since there are $\dfrac{n\,(n-1)}{2}$ such products. The sum of the distances from the diametral conic $= \dfrac{2}{n}\,\Sigma y$. The mean distance is then the same for both curves, since there are two such distances in the one case, and n in the other; and the two curves have the same diameter.

54. There is no difficulty in seeing that a curve of the n^{th} degree may have other *curvilinear diameters* of any degree up to the $(n-1)^{st}$. Thus the locus of a point such that the sum of the products in threes of its distances from the curve should vanish, is

found by putting the coefficient of ρ^{n-3} in the transformed equation $= 0$; and since this coefficient involves no higher than the third powers of the variables, the locus will be of the third degree. We may see too, in like manner, that

$$\Sigma(y - y_1)(y - y_2)(y - y_3) = \frac{n(n-1)(n-2)}{1.2.3} y^3 - \frac{(n-1)(n-2)}{1.2} y^2 \Sigma(y) + \frac{n-2}{1} y \Sigma(y_1 y_2) - \Sigma(y_1 y_2 y_3),$$

and we can readily infer hence that the curve and its cubical diameter will have the same mean distance, mean product in pairs, and mean product in threes of the distances. So in like manner for diameters of higher dimensions. More light will be thrown on the subject of these curvilinear diameters by considerations which we shall explain presently.

55. To the mention we have made of diameters, we may add some notice of centres. If all the terms of the degree $n-1$ were wanting in the equation, then the algebraic sum of all the radii vectores through the origin would vanish, and the origin might in one sense be called a centre.

The name *centre*, however, is ordinarily only applied to the case where every value of the radius vector is accompanied by an equal and opposite one. In this case, if the equation be transformed to polar co-ordinates, it must be a function of ρ^2 only. If the curve then be of an even degree, its equation in x and y, referred to the centre, can contain none of the odd powers of the variables, and must be of the form

$$u_0 + u_2 + u_4 + \&c. = 0.$$

If the curve be of an odd degree, its polar equation must be reducible to a function of ρ^2 by dividing by ρ; and the x and y equation can contain none of the even powers of the variables, but must be of the form

$$u_1 + u_3 + u_5 + \&c. = 0.$$

It is plain (from Art. 42) that if a curve of an odd degree have a centre, that centre must be a point of inflexion. It is also evident that it is only in exceptional cases that a curve of any degree above the second will have a centre; since it is not generally pos-

sible, by transformation of co-ordinates, to remove so many terms from the equation as to bring it to either of the forms given above.

POLES AND POLARS.

56. We pass now to an important theorem, first given by Cotes in his Harmonia Mensurarum: *If on each radius vector, through a fixed point* O, *there be taken a point* R *such that*

$$\frac{n}{\mathrm{OR}} = \frac{1}{\mathrm{OR}_1} + \frac{1}{\mathrm{OR}_2} + \frac{1}{\mathrm{OR}_3} + \&c.,$$

then the locus of R *will be a right line.*

For, making O the origin, the equation which determines OR_1, &c. is of the form

$$\mathrm{A} \cdot \frac{1}{\rho^n} + (\mathrm{B}\cos\theta + \mathrm{C}\sin\theta)\frac{1}{\rho^{n-1}}$$
$$+ (\mathrm{D}\cos^2\theta + \mathrm{E}\cos\theta\sin\theta + \mathrm{F}\sin^2\theta)\frac{1}{\rho^{n-2}} + \&c. = 0.$$

Hence
$$\frac{n}{\mathrm{OR}} = -\frac{(\mathrm{B}\cos\theta + \mathrm{C}\sin\theta)}{\mathrm{A}},$$

or, returning to x and y co-ordinates,

$$\mathrm{B}x + \mathrm{C}y + n\mathrm{A} = 0.$$

This line, being analogous to the polar in the case of conics, which has been proved to be the locus of harmonic means of radii vectores drawn from a given point (*Conics*, p. 134), we shall extend the use of the words *pole* and *polar*, and call this the *polar line of the origin.*

57. We may form the equations of polar curves of higher orders just as we formed the equations of diametral curves. Thus we may form the equation

$$\Sigma\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_2}\right) = 0,$$

which (as at Art. 53) we can see is equivalent to

$$\frac{n(n-1)}{2}\frac{1}{\rho^2} - (n-1)\frac{1}{\rho}\,\Sigma\left(\frac{1}{\rho}\right) + \Sigma\left(\frac{1}{\rho_1\rho_2}\right) = 0,$$

and represents a conic such that the harmonic mean of the distances of the origin from the conic is equal to the harmonic mean

of its distances from the curve; and the reciprocal of the product of its distances from the conic is equal to the mean product in pairs of the reciprocals of its distances from the curve. This conic we shall call *the polar conic* of the origin. Its equation, found by putting in, for $\Sigma\left(\frac{1}{\rho}\right)$, $\Sigma\left(\frac{1}{\rho_1\rho_2}\right)$, their values from the equation of the curve, is

$$\frac{n(n-1)}{2}\frac{1}{\rho^2} + (n-1)\frac{B\cos\theta + C\sin\theta}{A}\frac{1}{\rho} + \frac{D\cos^2\theta + E\sin\theta\cos\theta + F\sin^2\theta}{A} = 0,$$

which is equivalent to

$$\frac{n(n-1)}{2}u_0 + (n-1)u_1 + u_2 = 0.$$

58. And so in like manner we can obtain the polar curve of any higher order, k, by forming the equation

$$\Sigma\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_2}\right)(\&c.)\left(\frac{1}{\rho} - \frac{1}{\rho_k}\right) = 0,$$

which is equivalent to

$$\frac{n(n-1)\ldots(n-k+1)}{1.2\ldots k}\left(\frac{1}{\rho}\right)^k - \frac{(n-1)\ldots(n-k+1)}{1\ldots(k-1)}\left(\frac{1}{\rho}\right)^{k-1}\Sigma\frac{1}{\rho} + \frac{(n-2)\ldots(n-k+1)}{1\ldots(k-2)}\left(\frac{1}{\rho}\right)^{k-2}\Sigma\frac{1}{\rho_1\rho_2}, \ \&c. = 0,$$

a curve of the k^{th} degree, which possesses the property (if OR denote a radius vector to the curve, and Or to the polar curve),

$$\frac{1}{n}\Sigma\frac{1}{OR} = \frac{1}{k}\Sigma\frac{1}{Or},$$

$$\frac{1.2}{n(n-1)}\Sigma\frac{1}{OR_1 . OR_2} = \frac{1.2}{k(k-1)}\Sigma\frac{1}{Or_1\, Or_2},$$

$$\frac{1.2.3}{n(n-1)(n-2)}\Sigma\frac{1}{OR_1 . OR_2 . OR_3} = \frac{1.2.3}{k(k-1)(k-2)}\Sigma\frac{1}{Or_1 . Or_2 . Or_3},$$

&c.

The equation of the polar curve is obtained by putting in the values for $\Sigma\frac{1}{\rho}$, &c., and is

$$\left(\frac{1}{\rho}\right)^k + \frac{k}{n}\left(\frac{1}{\rho}\right)^{k-1}\left(\frac{B\cos\theta + C\sin\theta}{A}\right)$$
$$+ \frac{k(k-1)}{n(n-1)}\left(\frac{1}{\rho}\right)^{k-2}\frac{D\cos^2\theta + E\cos\theta\sin\theta + F\sin^2\theta}{A} + \&c. = 0;$$

or
$$u_0 + \frac{k}{n}u_1 + \frac{k(k-1)}{n(n-1)}u_2 + \frac{k(k-1)(k-2)}{n(n-1)(n-2)}u_3 + \&c. = 0.$$

It is plain, from the manner in which the equations of these polar curves have been formed, that the polar line of the origin is the same with regard to all these polar curves, for the harmonic mean of the radii vectores is the same for all the curves; that the polar conic of the origin is the same for all these curves which are above the second degree, since the mean value of the product in pairs of the reciprocal of the distances from the origin is the same for all these curves; and, generally, that any of the polar curves of the origin is also its polar curve with regard to the other polar curves of higher degree.

59. Let us trace now the forms which the relations just given assume when the point O is at an infinite distance. The equation which determines the polar line,

$$\Sigma\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = 0, \text{ or } \Sigma\left(\frac{1}{OR} - \frac{1}{OR_1}\right) = 0,$$

where R is the point on the polar, R_1 one of the points on the curve, is equivalent to

$$\Sigma\left(\frac{RR_1}{OR.OR_1}\right) = 0;$$

but if O be at an infinite distance,

$$OR = OR_1 = OR_2 = \&c.,$$

and the denominators in all the fractions may be considered as the same; the condition then becomes $\Sigma(RR_1) = 0$; the sum vanishes of the intercepts between the polar and the curve, on the parallel chords which meet at O; and we learn that *the polar of a point at an infinite distance is the diameter of the system of parallel chords which are directed to that infinitely distant point.*

And so for the polar conic. The equation which determines it,

$$\Sigma\left(\frac{1}{\rho}-\frac{1}{\rho_1}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_2}\right)=0,\ \text{or}\ \Sigma\left(\frac{1}{\text{OR}}-\frac{1}{\text{OR}_1}\right)\left(\frac{1}{\text{OR}}-\frac{1}{\text{OR}_2}\right)=0,$$

becomes

$$\Sigma\left(\frac{\text{RR}_1}{\text{OR}\cdot\text{OR}_1}\cdot\frac{\text{RR}_2}{\text{OR}.\text{OR}_2}\right)=0,$$

and when O is infinitely distant, reduces to $\Sigma\,\text{RR}_1.\text{RR}_2=0$, or $\Sigma\,(y-y_1)\,(y-y_2)=0$, the very equation which determines the diametral conic. And so, in general, *the curvilinear diameter of any order is identical with the polar curve of the same order, of the infinitely distant point on the system of parallel chords to which the given diametral curve corresponds.*

60. Mac Laurin has given a theorem, which is the extension of Newton's theorem (Art. 52): "*If through any point* O *a line be drawn meeting the curve in n points, and at these points tangents be drawn, and if any other line through* O *cut the curve in* R_1, R_2, &c., *and the system of n tangents in* r_1, r_2, *&c., then* $\Sigma\,\dfrac{1}{\text{OR}}=\Sigma\,\dfrac{1}{\text{O}r}$."

It is evident that two points determine the polar line; that, therefore, if two lines through O meet two curves in the same points, OR_1, OR_2, &c., OS_1, OS_2, &c., the polar of O, with regard to both curves, must be the same, since two points of it, R and S, are the same for both. This will be equally true if the two lines OR, OS coincide, that is to say:—"If two curves of the n^{th} degree touch each other at n points in a right line, then the polar of any point on that right line will be the same for both curves; and therefore if any radius vector through such a point meet both curves, we must have $\Sigma\,\dfrac{1}{\text{OR}}=\Sigma\,\dfrac{1}{\text{O}r}$."

61. We have hitherto only showed how to form the equations of the polar curves of the origin; we proceed now to show how to form the equations of the polar curves of any given point. We might obtain them by transformation of co-ordinates, from those already given, but we prefer the following method, as more symmetrical, and more convenient for future applications. We shall use trilinear co-ordinates, x, y, z.

We have seen that the equation of the polar line is found by

forming the equation $\Sigma\left(\frac{1}{\rho}-\frac{1}{\rho_1}\right)=0$, or putting $\rho=\mathrm{OR}$, $\rho_1=\mathrm{OR}_1$, and suppressing the factor OR, which occurs in all the denominators, $\Sigma\left(\frac{\mathrm{RR}_1}{\mathrm{OR}_1}\right)=0$. The equation of the polar conic is found by forming the equation

$$\Sigma\left(\frac{1}{\rho}-\frac{1}{\rho_1}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_2}\right)=\Sigma\left(\frac{\mathrm{RR}_1}{\mathrm{OR}_1}\cdot\frac{\mathrm{RR}_2}{\mathrm{OR}_2}\right)=0,$$

and so for the rest. If then we could form the equation whose roots are $\frac{\mathrm{RR}_1}{\mathrm{OR}_1}$, $\frac{\mathrm{RR}_2}{\mathrm{OR}_2}$, &c., viz., the ratios in which the line OR is cut at the n points where the curve meets it, the coefficient of the second term of this equation, being equal to the sum of these ratios, when put $=0$, would give the equation of the polar line; the coefficient of the next term, being equal to the sum of the products in pairs, would give the equation of the polar conic; and so on.

But such an equation can readily be formed; for we have seen (*Conics*, p. 5) that if the co-ordinates of O be $x_, y_, z_,$ of R, $x\,y\,z$ (both being for the present supposed known), the co-ordinates of the point R_1 cutting the line joining them in the ratio $\lambda:\mu\left(\frac{\mathrm{RR}_1}{\mathrm{OR}_1}=\frac{\mu}{\lambda}\right)$ are $\frac{\mu x_,+\lambda x}{\mu+\lambda}$, $\frac{\mu y_,+\lambda y}{\mu+\lambda}$, $\frac{\mu z_,+\lambda z}{\mu+\lambda}$. Now, by hypothesis, the point R_1 is on the curve; these values of the co-ordinates must then satisfy the equation of the curve. And since the equation in xyz is homogeneous, we may suppress the common denominator $\mu+\lambda$, and we learn that if in the equation of a curve $\phi(x,y,z)=0$ we substitute for x, y, z, $\mu x_,+\lambda x$, $\mu y_,+\lambda y$, $\mu z_,+\lambda z$, we shall have an equation of the n^{th} degree in $\mu:\lambda$, the roots of which give the ratios in which the line OR is cut at each of the n points, where it meets the curve; and these n values for $\mu:\lambda$, being substituted in the co-ordinates of R_1 just given, give the co-ordinates of the n points where OR meets the curve. But if now we suppose x, y, z, the co-ordinates of R, to be variable, the coefficients of the equation in $\mu:\lambda$, put $=0$, give the equations of the several polar curves of the point O.

62. The differential calculus enables us simply to write down

the result of this substitution. If there be any function of three variables, then, by Taylor's theorem,

$$\phi(x+h,\ y+k,\ z+l) = \phi(x, y, z) + \left(h\frac{d\phi}{dx} + k\frac{d\phi}{dy} + l\frac{d\phi}{dz}\right)$$
$$+\frac{1}{1.2}\left(h^2\frac{d^2\phi}{dx^2} + k^2\frac{d^2\phi}{dy^2} + l^2\frac{d^2\phi}{dz^2} + 2hk\frac{d^2\phi}{dxdy} + 2kl\frac{d^2\phi}{dydz} + 2lh\frac{d^2\phi}{dzdx}\right)$$
$$+ \&c.$$

The result then of substituting $\lambda x + \mu x_,$ &c. for x, y, z, is found by writing $\frac{\mu x_,}{\lambda}$ for h, $\frac{\mu y_,}{\lambda}$ for k, $\frac{\mu z_,}{\lambda}$ for l, in the preceding equation. If the equation of the curve be $U = 0$, the transformed equation $[U] = 0$, we have

$$[U] = \lambda^n U + \lambda^{n-1}\mu\left(x_,\frac{dU}{dx} + y_,\frac{dU}{dy} + z_,\frac{dU}{dz}\right)$$
$$+\frac{\lambda^{n-2}\mu^2}{1.2}\left(x_,^2\frac{d^2U}{dx^2} + y_,^2\frac{d^2U}{dy^2} + z_,^2\frac{d^2U}{dz^2}\right.$$
$$\left. + 2x_,y_,\frac{d^2U}{dxdy} + 2y_,x_,\frac{d^2U}{dydz} + 2z_,x_,\frac{d^2U}{dzdx}\right)$$
$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$+\frac{\mu^{n-2}\lambda^2}{1.2}\left\{x^2\left(\frac{d^2U}{dx^2}\right)_, + y^2\left(\frac{d^2U}{dy^2}\right)_, + z^2\left(\frac{d^2U}{dz^2}\right)_,\right.$$
$$\left. + 2xy\left(\frac{d^2U}{dxdy}\right)_, + 2yz\left(\frac{d^2U}{dydz}\right)_, + 2zx\left(\frac{d^2U}{dzdx}\right)_,\right\}$$
$$+\mu^{n-1}\lambda\left\{x\left(\frac{dU}{dx}\right)_, + y\left(\frac{dU}{dy}\right)_, + z\left(\frac{dU}{dz}\right)_,\right\} + \mu^n U_,.$$

We have added the closing terms of the expansion, which must evidently be symmetrical with the commencing terms. $\left(\frac{dU}{dx}\right)_,$ &c. denotes the result of substituting $x_, y_, z_,$ for $x\ y\ z$ in $\frac{dU}{dx}$, &c.

This expansion may be written in an abbreviated form by the use of symbols of operation. Thus if Δ denote the operation

$$x_,\frac{d}{dz} + y_,\frac{d}{dy} + z_,\frac{d}{dz},$$

then this operation, twice repeated,

$$\Delta^2 = \left(x_, \frac{d}{dx} + y_, \frac{d}{dy} + z_, \frac{d}{dz}\right)^2 = x_,^2 \frac{d^2}{dx^2} + y_,^2 \frac{d^2}{dy^2} + z_,^2 \frac{d^2}{dz^2}$$
$$+ 2x_,y_, \frac{d^2}{dxdy} + 2y_,z_, \frac{d^2}{dydz} + 2z_,x_, \frac{d^2}{dzdx}.$$

In like manner we can form

$$\Delta^3 = \left(x_, \frac{d}{dx} + y_, \frac{d}{dy} + z_, \frac{d}{dz}\right)^3 \text{ \&c.}$$

In order to distinguish the co-ordinates of the points which enter into these formulæ, we shall use the following notation. We shall write

$$\Delta_, \mathrm{U} = x_, \frac{d\mathrm{U}}{dx} + y_, \frac{d\mathrm{U}}{dy} + z_, \frac{d\mathrm{U}}{dz},$$

$$\Delta \mathrm{U}_, = x \left(\frac{d\mathrm{U}}{dx}\right)_, + y \left(\frac{d\mathrm{U}}{dy}\right)_, + z \left(\frac{d\mathrm{U}}{dz}\right)_,,$$

where the suffix attached to Δ refers to the co-ordinates which multiply the differential coefficients; but the suffix attached to U denotes that the co-ordinates $x_, y_, z_,$ are to be substituted in $\frac{d\mathrm{U}}{dx}$, &c.

So $\Delta_1 \mathrm{U}_2$ would denote $x_1 \left(\frac{d\mathrm{U}}{dx}\right)_2 + y_1 \left(\frac{d\mathrm{U}}{dy}\right)_2 + z_1 \left(\frac{d\mathrm{U}}{dz}\right)_2$, or the result of substituting the co-ordinates $x_2y_2z_2$ in $\Delta_1 \mathrm{U}$. $\Delta_2 \mathrm{U}_1$ would denote the result of substituting the same co-ordinates for xyz in $\Delta \mathrm{U}_1$. The reader will have no difficulty in seeing what meaning is to be attached to the symbols $\Delta^2 \mathrm{U}_1$ and $\Delta^2{}_1 \mathrm{U}$, &c. The result of the substitution may then be written

$$\lambda^n \mathrm{U} + \lambda^{n-1} \mu (\Delta_1 \mathrm{U}) + \frac{\lambda^{n-2} \mu^2}{1.2} (\Delta^2{}_1 \mathrm{U}) + \frac{\lambda^{n-3} \mu^3}{1.2.3} (\Delta^3{}_1 \mathrm{U}) + \ldots\ldots\ldots$$
$$+ \mu^n \mathrm{U}_1 + \mu^{n-1} \lambda (\Delta \mathrm{U}_1) + \frac{\mu^{n-2} \lambda^2}{1.2} (\Delta^2 \mathrm{U}_1) + \frac{\mu^{n-3} \lambda^3}{1.2.3} (\Delta^3 \mathrm{U}_1) + \text{\&c.} = 0.$$

This equation, then, if the points O $(x_,y_,z_,)$ and R (xyz) be known, gives us the co-ordinates of the n points, where OR meets the curve, viz., $\frac{\lambda_, x + \mu_, x_,}{\lambda_, + \mu_,}$, $\frac{\lambda_, y + \mu_, y_,}{\lambda_, + \mu_,}$, $\frac{\lambda_, z + \mu_, z_,}{\lambda_, + \mu_,}$, where $\lambda_, : \mu_,$ is any of the roots of the equation just written.

63. Since $\frac{\mu}{\lambda} = \frac{RR_1}{OR_1}$, and since the product of all the roots $\frac{\mu}{\lambda} = \frac{U}{U_1}$, we have

$$\frac{RR_1 . RR_2 . RR_3 \ \&c.}{OR_1 . OR_2 . OR_3 \ \&c.} = \frac{U}{U_1};$$

or "the continued product of the distances measured on a given line of any point from the curve is proportional to the result of substituting the co-ordinates of that point in the equation of the curve." We are thus led again to the theorem proved already (Art. 46).

Since by the theory of equations $\Sigma\left(\frac{\mu}{\lambda}\right) = \frac{\Delta U_1}{U_1}$, and since (as we showed, Art. 61) the equation of the polar line is found by putting $\Sigma \frac{RR_1}{OR_1} = 0$, the equation of the polar line of the point x, y, z, is

$$\Delta U_1 = x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0.$$

Similarly the equation of the polar conic of the same point is

$$\Delta^2 U_1 = x^2\left(\frac{d^2U}{dx^2}\right)_1 + y^2\left(\frac{d^2U}{dy^2}\right)_1 + z^2\left(\frac{d^2U}{dz^2}\right)_1$$
$$+ 2xy\left(\frac{d^2U}{dxdy}\right)_1 + 2yz\left(\frac{d^2U}{dydz}\right)_1 + 2zx\left(\frac{d^2U}{dzdx}\right)_1 = 0.$$

And so in like manner we can write down the equations of the polar curves of higher dimensions.

Since we have written down the closing terms of the expansion, which must be symmetrical with the commencing terms, it appears that the equation of the polar curve of the $(n-1)^{st}$ degree may be written at pleasure in either of the forms

$$\Delta^{n-1} U_1 = 0, \ \text{or} \ \Delta_1 U = x_1 \frac{dU}{dx} + y_1 \frac{dU}{dy} + z_1 \frac{dU}{dz} = 0.$$

So the polar curve of the $(n-2)^{nd}$ degree may have its equation written in either of the forms $\Delta^{n-2} U_1 = 0$, or $\Delta^2{}_1 U = 0$; and the polar line itself may have its equation written in the form

$$\Delta_1^{n-1} U = 0.$$

The polar curve of the $(n-1)^{st}$ degree, whose equation is found by performing once the operation $\Delta_1 U$, we shall call the first polar; that of the $(n-2)^{nd}$ degree, whose equation is found by twice performing the same operation, we shall call the second polar; and so on.

From the manner in which these equations are formed, it is manifest, as was remarked before, that the polar curve of any degree is also the polar curve of the point O, with regard to all its polar curves of a degree higher than its own. This follows at once from the equation

$$\Delta_{\prime}^{k}(\Delta_{\prime}^{l}U) = \Delta_{\prime}^{k+l}U.$$

64. *The locus of all the points whose polar lines pass through a given point is the first polar of that point.*

The equation

$$x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0$$

expresses a relation between xyz, the co-ordinates of any point on the polar line, and $x_1y_1z_1$ those of the pole. If, therefore, the first point be fixed $x_2y_2z_2$, and the second variable, the locus of the latter point must be

$$x_2\left(\frac{dU}{dx}\right) + y_2\left(\frac{dU}{dy}\right) + z_2\left(\frac{dU}{dz}\right) = 0.$$

Hence *every right line has* $(n-1)^2$ *poles.* For, take any two points on it, the poles of the right line must lie on the first polars of each of these points; therefore they are the intersections of these curves. Also, *the first polars of all the points of a right line pass through the same* $(n-1)^2$ *points;* viz., through the $(n-1)^2$ poles of the right line.

In like manner the locus of points whose polar conics pass through a given point is the second polar of the point; and so on.

65. *The polar line of any point on the curve is the tangent at the point.*

For we have seen (Art. 56) that the equation of the polar line is in general $u_1 + nu_0 = 0$, which, when the origin is on the curve, reduces to $u_1 = 0$, the equation of the tangent (Art. 29.) Hence, by the last Article, *the points of contact of tangents which pass through a given point lie on the first polar of that point.* Con-

versely, too, if the polar of a point pass through the point, that point is on the curve, since we must have $u_0 = 0$.

All the other polars of a point on the curve touch the curve at that point. For, the general formula for a polar curve,

$$\left(x_1 \frac{d}{dx} + y_1 \frac{d}{dy} + z_1 \frac{d}{dz}\right)^k \mathrm{U} = 0,$$

reduces for the origin, whose $x_1 = 0$, $y_1 = 0$, to $\frac{d^k\mathrm{U}}{dz^k}$; but the equation being $u_1 z^{n-1} + u_2 z^{n-2} + \&c. = 0$, it is plain that no matter how we differentiate with respect to z, u_1 still represents the lowest terms in x and y.

66. *If a curve have a multiple point of the degree k, that point will be a multiple point of the degree $k-1$ on every first polar; of the degree $k-2$ on every second polar; and so on.*

For if the origin be at the multiple point, the lowest terms in x and y will be of the degree k; in the first polar,

$$x_1 \frac{d\mathrm{U}}{dx} + y_1 \frac{d\mathrm{U}}{dy} + z_1 \frac{d\mathrm{U}}{dz} = 0,$$

the lowest terms will be of the degree $k - 1$, and therefore the origin will be a multiple point of that order; the equation of the second polar, involving the second differentials of the equation of the curve, will contain x and y at lowest in the degree $(k - 2)$, and so on.

If two tangents at the multiple point on the curve coincide, the coincident tangent will be a tangent to the first polar. For u_k is of the form a^2bcd, &c., and therefore both $\frac{du_k}{dx}$ and $\frac{du_k}{dy}$ contain a as a factor; and therefore the lowest terms in the equation of the polar, $x_1 \frac{du_k}{dx} + y_1 \frac{du_k}{dy}$ contain a as a factor; $z_1 \frac{du}{dz}$ plainly contains no terms below the degree k in x and y. And, in general, if l tangents to the multiple point on the curve coincide, $l - 1$ of them will be coincident tangents at the multiple point on the first polar; $l-2$ at the multiple point on the second polar; and so on. For if u_k have any factor in the l^{th} degree, that factor will be of the $(l - 1)^{st}$ degree in all the first differentials of u_k; it will enter in the $(l - 2)^{nd}$ degree into all the second differentials of u_k, &c.

67. If a curve have a double point it is easy to construct the tangent at that point to the first polar of any other point. For, let the equation of the curve be

$$xy + u_3 + u_4 + \&c. = 0.$$

Then the lowest terms in the equation

$$x_1 \frac{dU}{dx} + y_1 \frac{dU}{dy} + z_1 \frac{dU}{dz} = 0$$

will be

$$x_1 y + y_1 x = 0.$$

Now $x_1 y - y_1 x$ represents the line joining the double point to the point whose first polar we wish to describe; the tangent required will then be the fourth harmonic, to this line and x and y the two tangents at the double point.

When the two tangents coincide, then the tangent to every first polar coincides with these tangents.

SECT. IV.—GENERAL THEORY OF MULTIPLE POINTS AND TANGENTS.

68. We proceed now to investigate the conditions in general that a curve should have multiple points or tangents, and to lay down rules by which to find their position if they exist. We shall follow exactly the same course of investigation as that already employed in the case of the origin. We shall consider a series of radii vectores drawn through a given point; we shall form the equation which determines the co-ordinates of the n points where any such radius vector meets the curve, and we shall examine the conditions that one or more of these points should coincide with the given point itself. We shall use trilinear co-ordinates, and follow the same method of investigation as that employed in Art. 61.

Given then the co-ordinates of two points, $x_1y_1z_1$, $x_2y_2z_2$, the co-ordinates of any point where the line joining them meets the curve are proportional to $\lambda x_1 + \mu x_2$, $\lambda y_1 + \mu y_2$, $\lambda z_1 + \mu z_2$, where $\lambda : \mu$ is one of the roots of the equation

$$\lambda^n U_1 + \lambda^{n-1}\mu \, \Delta_2 U_1 + \frac{\lambda^{n-2}}{1.2} \mu^2 \Delta^2{}_2 U_2 + \ldots$$

$$+ \mu^n U_2 + \lambda \mu^{n-1} \, \Delta_1 U_2 + \lambda^2 \frac{\mu^{n-2}}{1.2} \, \Delta^2{}_1 U_1 + \&c. = 0.$$

In order that one of these points should coincide with $x_1y_1z_1$, it is necessary that one of the roots of this equation should be $\mu = 0$. But this clearly will not be the case unless $U_1 = 0$: and it is otherwise evident that the condition that $x_1y_1z_1$ should be on the curve is, that its co-ordinates, when substituted in the equation of the curve, should satisfy it.

69. Two of the points in which the line meets the curve will coincide with $x_1y_1z_1$, if the above equation be divisible by μ^2; that is, if not only $U_1 = 0$, but also $\Delta_2 U_1 = 0$: now it is plain that if the line joining $x_2y_2z_2$ to $x_1y_1z_1$ (a point on the curve) meet the curve in two points which coincide with $x_1y_1z_1$, then $x_2y_2z_2$ must lie on the tangent (or tangents, if possible) which can be drawn to the curve at $x_1y_1z_1$: but we have now proved that in this case $x_2y_2z_2$ must satisfy the equation

$$\Delta_2 U_1 = 0, \text{ or } x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0.$$

Hence at a given point on the curve there is but one tangent, whose equation is that just written. It appears, too, that the polar line of a point on the curve is the tangent, as we have seen already.

70. The same equation would enable us to find the point of contact of tangents drawn through a given point. Were we given the point $x_2y_2z_2$, then the point of contact $x_1y_1z_1$ must satisfy the equation

$$x_2\frac{dU}{dx} + y_2\frac{dU}{dy} + z_2\frac{dU}{dy} = 0.$$

Hence *the points of contact of tangents which can be drawn from a given point to a curve of the n^{th} degree lie on a curve of the $n-1^{st}$ degree:* viz., on the first polar of $x_2y_2z_2$, with regard to the given curve, as has been proved in Art. 65.

The curve and its first polar must clearly intersect in $n(n-1)$ points; and since at each of these intersections the relations $U_1 = 0$, $\Delta_2 U_1 = 0$ will be satisfied, we see that *from a given point there can be drawn $n(n-1)$ tangents to a curve of the n^{th} degree.* Or again (*Conics*, p. 253), *the degree of the reciprocal of a curve of the n^{th} degree is in general $n(n-1)$.*

71. If, however, the curve have a double point, it was proved (Art. 66) that the first polar of any given point must pass through that double point. The double point, therefore (see note, p. 31), counts for two among the intersections of the curve with its first polar. But the line joining the point $x_2y_2z_2$ to the double point is not a tangent in the ordinary sense of the word, though it is indeed included among the solutions to the problem we have been discussing (viz., to draw a line through $x_2y_2z_2$, so as to meet the curve in two coincident points); for we have shown that *every* line through the double point must be considered as there meeting the curve in two coincident points. Now the entire number of solutions to this problem being always $n(n-1)$ (viz., the intersections of U and ΔU), the number of tangents, properly so called, which can be drawn to the curve is diminished by two for every double point on the curve; or *the degree of the reciprocal of a curve of the n^{th} degree having δ double points is* $n(n-1) - 2\delta$.

72. If the curve have a cusp, we have proved (Art. 66) that the first polar not only passes through the cusp, but also has its tangent the same with the tangent at the cusp. Hence (see note, p. 31) this cusp counts as three among the intersections of the curve with its first polar, and the remaining intersections are consequently diminished by three for every cusp on the curve. Hence *the degree of the reciprocal of a curve having δ ordinary double points and κ cusps, is*

$$n(n-1) - 2\delta - 3\kappa.*$$

73. The same principles would show the effect of any higher multiple point on the degree of the reciprocal. A multiple point of the order k would (Art. 66) be a multiple point of the order

* According to M. Poncelet, Waring was the first who investigated the problem of the number of tangents which can be drawn from a given point to a curve of the n^{th} degree. (Miscellanea Analytica, p. 100.) This number he fixed as at most n^2. M. Poncelet showed (Gergonne's Annales, vol. viii. p. 213) that this limit was fixed too high; that the points of contact lie on a curve of the $(n-1)^{st}$ degree, and that their number cannot exceed $n(n-1)$. Finally, M. Plücker pointed out the cases in which the number of these tangents is less than $n(n-1)$, and thereby fully explained (as we shall do further on) why it is that only n tangents can be drawn to the reciprocal of a curve of the n^{th} degree, though that reciprocal is, in general, of the degree $n(n-1)$.

$(k-1)$ on the first polar, and therefore the number of remaining intersections, and consequently the degree of the reciprocal would be diminished by $k(k-1)$.

The same result may be otherwise stated by the help of the following considerations. A system of k right lines has in general $\frac{k(k-1)}{2}$ double points, viz., the points in which each pair of lines intersect; but if the lines all pass through the same point, the double points disappear, and we have instead a multiple point of the order k. Thus we are led to the conception, *a multiple point of the order k arises from the union of* $\frac{k(k-1)}{2}$ *double points.*

And, since each double point diminishes the degree of the reciprocal by two, the result at which we have arrived may be enunciated:—*The effect of a multiple point of the order k, on the degree of the reciprocal, is the same as that of the equivalent number of double points.*

If l tangents at the multiple point coincide, it was proved (Art. 66) that $l-1$ of them would be coincident tangents to the first polar; we should therefore add $(l-1)$ to the number by which the multiple point would otherwise diminish the degree of the reciprocal.

74. We return now to the discussion of the equation

$$\lambda^n U_1 + \lambda^{n-1}\mu \Delta_2 U_1 + \frac{\lambda^{n-2}\mu^2}{1.2}\Delta^2{}_2 U_1 + \&c. = 0.$$

We have already examined the conditions that a line drawn through $x_1y_1z_1$ should meet the curve in two coincident points. If, however, $x_1y_1z_1$ were such as to make

$$\left(\frac{dU}{dx}\right)_1 = 0,\ \left(\frac{dU}{dy}\right)_1 = 0,\ \left(\frac{dU}{dz}\right)_1 = 0,$$

then
$$\Delta_2 U_1 = x_2\left(\frac{dU}{dx}\right)_1 + y_2\left(\frac{dU}{dy}\right)_1 + z_2\left(\frac{dU}{dz}\right)_1 = 0$$

would be satisfied, whatever were $x_2y_2z_2$. The point $x_1y_1z_1$ would then be a double point, and *every* line drawn through it would meet the curve in two coincident points.

We see then that the curve expressed by the general equation

in Cartesian or trilinear co-ordinates will not have any double points unless the coefficients be connected by a certain relation. For the three curves, $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$, will not, in general, have any point common to all three, and therefore their equations cannot all be made to vanish together. If between these three equations we eliminate xyz, we shall have a relation between the coefficients, which will be the condition that these three polars should intersect, or that the curve U should have a double point. Thus we found the condition that the conic,

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0,$$

should have a double point (or should break up into right lines), by eliminating xyz between

$$\begin{aligned} Ax + By + Dz &= 0, \\ Bx + Cy + Ez &= 0, \\ Dx + Ey + Fz &= 0, \end{aligned}$$

and we found

$$AE^2 + CD^2 + FB^2 - ACF - 2BDE = 0.$$

In the general case the resulting condition will be of the degree $3(n-1)^2$ in the coefficients of the given equation, for (see Note on Elimination at the end of the volume) since the three equations $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$ are each of the degree $(n-1)$, the coefficients of *each* enter into the final result in the degree $(n-1)^2$; but the coefficients in $\frac{dU}{dx}$, &c., are each of the first degree in the coefficients of the general equation. The final result is therefore of the degree $3(n-1)^2$.

75. We pass now to the conditions that a line through $x_1y_1z_1$ should meet the curve in three consecutive points. They are evidently $U_1 = 0$, $\Delta_2 U_1 = 0$, $\Delta^2{}_2 U_1 = 0$. It appears, then, that if $x_1y_1z_1$ be a double point, the equation of the pair of tangents at it is

$$\Delta^2 U_1 = 0, \text{ or } x^2\left(\frac{d^2U}{dx^2}\right)_1 + 2xy\left(\frac{d^2U}{dxdy}\right)_1 + y^2\left(\frac{d^2U}{dy^2}\right)_1$$
$$+ 2xz\left(\frac{d^2U}{dxdz}\right)_1 + 2yz\left(\frac{d^2U}{dydz}\right)_1 + z^2\left(\frac{d^2U}{dz^2}\right)_1 = 0.$$

As we shall frequently have occasion to speak of the second differential coefficients which enter into this equation, we shall, for brevity, write

$$A = \frac{d^2U}{dx^2},\ B = \frac{d^2U}{dxdy},\ C = \frac{d^2U}{dy^2},\ D = \frac{d^2U}{dxdz},\ E = \frac{d^2U}{dydz},\ F = \frac{d^2U}{dz^2},$$

so as to write the preceding equation in the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0.$$

It is easy to verify that this equation breaks up into two right lines at a double point, for we have, by Euler's theorem of homogeneous functions (Lacroix, Traité Elementaire de Calcul. Differentiel., p. 411),

$$\begin{aligned} Ax_1 + By_1 + Dz_1 &= (n-1)\left(\frac{dU}{dx}\right)_1 = 0, \\ Bx_1 + Cy_1 + Ez_1 &= (n-1)\left(\frac{dU}{dy}\right)_1 = 0, \qquad \text{(A)} \\ Dx_1 + Ey_1 + Fz_1 &= (n-1)\left(\frac{dU}{dz}\right)_1 = 0, \end{aligned}$$

and eliminating $x_1y_1z_1$ between these equations, we have

$$AE^2 + CD^2 + FB^2 - ACF - 2BDE = 0,$$

which is precisely the condition that the equation written above should represent two right lines.

76. The double point will be a cusp, if the equation which represents the two tangents be a perfect square; that is, if

$$AC = B^2,\ AF = D^2,\ CF = E^2.$$

These three are only equivalent to one new condition, for if any one of them be satisfied, and the co-ordinates $x_1y_1z_1$ of the double point have any finite magnitude, the others must also be satisfied. For, solving for $x_1 : z_1$, $y_1 : z_1$, successively from each pair of the equations (A) of the last Article, we have

$$x_1 = \frac{CD - BE}{B^2 - AC} z_1 = \frac{E^2 - CF}{CD - BE} z_1 = \frac{BF - DE}{AE - BD} z_1,$$

$$y_1 = \frac{AE - BD}{B^2 - AC} z_1 = \frac{BF - DE}{CD - BE} z_1 = \frac{D^2 - AF}{AE - BD} z_1.$$

Hence, if $B^2 - AC = 0$, and neither $z_1 = 0$, nor x_1 or y_1 infinite,

we must have both numerator and denominator of every one of these fractions = 0.

77. The origin will be a triple point, if

$$A = 0,\ B = 0,\ C = 0,\ D = 0,\ E = 0,\ F = 0,$$

A, B, &c., being the second differential coefficients, as in the last Article. For then $\Delta^2{}_2 U_1 = 0$ whatever $x_2y_2z_2$, and it follows from equations (A), Art. 75, that $\left(\frac{dU}{dx}\right)_1 \left(\frac{dU}{dy}\right)_1 \left(\frac{dU}{dz}\right)_1$ each = 0, and therefore $\Delta_2 U_1 = 0$. Consequently *every* line through $x_2y_2z_2$ meets the curve in three coincident points. It is obvious that the three tangents at the triple point are given by the equation $\Delta^3 U_1 = 0$.

There is no difficulty in extending the same considerations to higher multiple points. $x_1y_1z_1$ is a multiple point of the order k, if all the differential coefficients of U of the order $k - 1$ vanish for that point, and the tangents at the multiple point are given by the equation $\Delta^k U_1 = 0$.

78. Thus we have shown how, by the help of the equation of Art. 68, to ascertain the existence, and discover the position of the multiple points of curves. We proceed now to show how, from the same equation, to discuss the ordinary and multiple tangents of the curve.

Let it be required to find the equation of all the tangents which can be drawn to a curve from a given point $x_1y_1z_1$. The line joining this point to any point on one of these tangents meets the curve in two consecutive points; but the equation which gives the points where this line meets the curve is (Art. 68)

$$\lambda^n U + \lambda^{n-1}\mu(\Delta_1 U) + \ldots\ldots + \mu^{n-1}\lambda(\Delta U_1) + \mu^n U_1 = 0,$$

which we shall, for shortness, write $Z = 0$. This equation in $\lambda : \mu$ must then have two equal roots. If we find, then, the condition that this should be the case by eliminating λ, μ between the equations $\frac{dZ}{d\lambda} = 0$, $\frac{dZ}{d\mu} = 0$, the result will be a relation which will be satisfied for the co-ordinates of every point on any tangent through $x_1y_1z_1$, and will therefore be the equation of all these tangents.

79. The reader will find the application of this method to curves of the second degree (*Conics*, p. 135). We shall add here, as a further illustration, the application of the same method to curves of the third degree.

The equation which gives the points where the line meets the curve which joins $x_1y_1z_1$ to a variable point xyz is

$$\lambda^3 U + \lambda^2\mu(\Delta_1 U) + \lambda\mu^2(\Delta U_1) + \mu^3 U_1 = 0.$$

The condition that this equation should have two equal roots is (see Note on Elimination)

$$(27UU_1^2 + 4\Delta^3 - 18\Delta\Delta_1 U_1)\, U = (\Delta^2 - 4\Delta_1 U_1)\,\Delta_1^2,$$

where, for brevity, we have written Δ instead of (ΔU_1), Δ_1 instead of $(\Delta_1 U)$.

In the preceding equation U, Δ_1, Δ are respectively of the third, second, and first degrees in xyz; the preceding equation, then, being of the sixth degree, shows that six tangents can be drawn from $x_1y_1z_1$ to the curve, as we know already.

The form of the equation shows that it represents a locus touching U in the points where U meets Δ_1. The other points, where U meets the locus, lie on the curve $\Delta^2 - 4\Delta_1 U_1 = 0$. Hence, *if from any point six tangents be drawn to a curve of the third degree, their six points of contact lie on a conic* (viz., $\Delta_1 = 0$), *and the six remaining points where these tangents meet the curve lie on another conic* (viz., $\Delta^2 - 4\Delta_1 U_1 = 0$), *and the form of the equation shows that these two conics have double contact with each other* (viz., in the points $\Delta\Delta_1$).

If the point $x_1y_1z_1$ lie on the curve, $U_1 = 0$; and to determine the two points in which any line through it meets the curve, we have the equation

$$\lambda^2 U + \lambda\mu\Delta_1 + \mu^2\Delta = 0.$$

Hence the equation of the tangents through the point is

$$\Delta_1^2 = 4U\Delta,$$

an equation of the fourth degree in xyz. Hence, *through a point on a curve of the third degree can in general be drawn but four tangents.* The tangent at the point, in fact, reckons as two.

80. And so in like manner in general. The condition that the equation

$$\lambda^n U + \lambda^{n-1}\mu\Delta_1 + \lambda^{n-2}\mu^2\Delta^2{}_1 + \ldots \lambda\mu^{n-1}\Delta + \mu^n U_1 = 0$$

should have two equal roots, is (see Note on Elimination) of the form

$$kU + (\Delta_1)^2\phi = 0,$$

where $\phi = 0$ is the condition that the equation deprived of its first term should have two equal roots. Hence the locus touches U at its points of intersection with Δ_1, as it plainly ought to do. Each of the $n(n-1)$ tangents meets the curve again in $n-2$ points, and it appears from the equation that these $n(n-1)(n-2)$ points lie on a curve ϕ of the degree $(n-1)(n-2)$. Moreover, ϕ is itself of the form $k'\Delta_1 + (\Delta^2{}_1)^2\psi$. Hence, the two curves, ϕ and Δ_1, touch each other at the points where the first and second polars of $x_1y_1z_1$ intersect.

81. There is no difficulty in seeing how these results are to be modified when $x_1y_1z_1$ is on the curve. The equation of the tangents is always of the form

$$kU_1 + (\Delta)^2\phi = 0,$$

where $\phi = 0$ is now the condition that the equation wanting its *last* term should have two equal roots. If $U_1 = 0$, the equation reduces to the square of the tangent Δ at the given point, and to $\phi = 0$, an equation of the $n^2 - n - 2$ degree, which represents the other tangents drawn from $x_1y_1z_1$ to the curve.

If the point $x_1y_1z_1$ were a double point, the last two terms of the equation Z would vanish. ϕ is, in general, of the form

$$k\Delta + (\Delta^2)^2\psi = 0.$$

If then $\Delta = 0$, the equation ϕ, which was already of the degree $n^2 - n - 2$, reduces to the square of the pair of tangents at the double point, and that of $n^2 - n - 6$ other tangents, which can be drawn from this point to the curve.

And so in like manner we can prove, that the number of tangents which can be drawn from a multiple point of the order k, is $n^2 - n - k(k+1)$.

The theory already given of the effect of multiple points upon the number of tangents which can be drawn from any point to a curve, shows that the condition that the equation Z in $\dfrac{\lambda}{\mu}$ should have two equal roots, which in general represents the $n(n-1)$ tan-

gents, will include as factors the square of the line joining the given point to every double point there may be on the curve, the cube of the line joining it to every cusp, the sixth power of the line to every triple point, &c., &c.

82. We proceed now to multiple tangents, and we shall commence by showing, that, though the curve expressed by the general equation we have been discussing will not in general have double points, it will ordinarily have double and stationary tangents. The abscissæ of the points where the curve is met by any line $y = ax + b$, are found by substituting this value for y in the equation of the curve. Now, since the equation of this line includes two arbitrary constants, we can determine a and b, so that the resulting equation shall fulfil any two conditions we please. With one constant at our disposal we could make the equation fulfil any one condition: for instance, have a pair of equal roots. The problem, "given a to determine b, so that the resulting equation should have a pair of equal roots," is no other than the problem to draw a tangent parallel to $y = ax$. With the two constants at our disposal we can either cause the resulting equation to have two distinct pairs of equal roots, or three roots, all equal to each other. The first problem is the problem of double tangents; the second, that of stationary tangents and points of inflexion. It would seem, then, that we may speak of double and stationary tangents as the ordinary singularities of curves, being such as all curves represented by the general equation will possess, except for particular values of the coefficients; and that all higher multiple tangents and all multiple points might be called the extraordinary singularities, being such as curves will not possess, except for particular values of the coefficients. But though the general equation in Cartesian or trilinear co-ordinates thus represents a curve having no double points, it is plain, in like manner, that the general equation in tangential co-ordinates represents a curve which has no points of inflexion or double tangents, but which, ordinarily, will have double points and cusps. We see, then, that double and stationary points, double and stationary tangents, are equally entitled to be ranked among the ordinary singularities of curves; being such, that if any curve possess the one, its reciprocal will possess the other.

83. Let us now endeavour, by the help of the equation (Z), to examine the double and stationary tangents of a curve. And we shall begin with the stationary tangents, whose points of contact are points of inflexion. We have proved already (Art. 75) that if $x_1y_1z_1$ be a point of inflexion, and xyz any point on its tangent which meets the curve in three consecutive points, we must have

$$U_1 = 0, \quad x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0,$$

$$x^2\left(\frac{d^2U}{dx^2}\right)_1 + y^2\left(\frac{d^2U}{dy^2}\right)_1 + z^2\left(\frac{d^2U}{dz^2}\right)_1 + 2xy\left(\frac{d^2U}{dxdy}\right)_1 + 2yz\left(\frac{d^2U}{dydz}\right)_1 + 2zx\left(\frac{d^2U}{dzdx}\right)_1 = 0.$$

The second of these is the equation of the tangent; but since the latter must also be satisfied for every point on the tangent, that equation of the second degree can be no other than the equation of the tangent multiplied by some linear factor. It follows, then, that this equation must be resolvable into two linear factors, and that, therefore, $x_1y_1z_1$ must satisfy the condition

$$\frac{d^2U}{dx^2}\left(\frac{d^2U}{dydz}\right)^2 + \frac{d^2U}{dy^2}\left(\frac{d^2U}{dxdz}\right)^2 + \frac{d^2U}{dz^2}\left(\frac{d^2U}{dxdy}\right)^2 - \frac{d^2U}{dx^2}\cdot\frac{d^2U}{dy^2}\cdot\frac{d^2U}{dz^2} - 2\frac{d^2U}{dydz}\cdot\frac{d^2U}{dzdx}\cdot\frac{d^2U}{dxdy} = 0.$$

We shall write this equation for shortness H(U), or sometimes simply $H = 0$.* Since every one of the quantities $\frac{d^2U}{dx^2}$, &c., involves the co-ordinates in the degree $n - 2$, the equation H represents a curve of the $3(n - 2)$ degree, which, of course, intersects the given curve in $3n(n - 2)$ points.

* The general theory of the number of points of inflexion on a curve was first given by M. Plücker. (See System der analytischen Geometrie, p. 264.) The use of the equation H, in determining the points of inflexion of a curve, was first pointed out by M. Hesse. He has given the name "functional determinant" to this condition, formed by eliminating xyz from the differentials with regard to x, y, z, of $\Delta^2U_1 = 0$. And it is obvious that such a condition can be formed from any homogeneous equation, whatever be the number of variables. Thus, if the equation were a function but of two variables, Hesse's functional determinant would be $\frac{d^2U}{dx^2}\,\frac{d^2U}{dy^2} = \left(\frac{d^2U}{dxdy}\right)^2$. If the equation be a function of four variables, and if we write, for brevity,

We must show that $H = 0$ is a condition for a point of inflexion not only necessary but sufficient; that is to say, that if the polar conic of any point on a curve break up into two right lines, that point will in general be a point of inflexion. But it has been shown (Art. 66) that the polar conic of a point on the curve touches the curve at that point; consequently, if the conic break up into two right lines, one of these right lines must be the tangent. We have therefore ΔU_1, a factor in $\Delta^2 U_1$, and, therefore, as was proved in the beginning of this Article, the line joining any point on ΔU_1 to $x_1 y_1 z_1$, meets the curve in three consecutive points.

It follows, then, that every one of the points of intersection of the curves U, H, will be in general a point of inflexion, and that, therefore, *a curve of the n^{th} degree has in general $3n(n-2)$ points of inflexion.*

84. If the curve have multiple points, however, the number of points of inflexion will be reduced. We have already seen (Art. 75) that the equation $H(U) = 0$ is satisfied for double points. H, in short, represents the locus of all points whose polar conics break up into two right lines; but this is true as well for double points as for points of inflexion, since the polar conic of a double point is the pair of tangents at that point. But we shall now show that the double point is also a double point on the curve H, and,

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$$a = \frac{d^2U}{dx^2}, \quad b = \frac{d^2U}{dy^2}, \quad c = \frac{d^2U}{dz^2}, \quad d = \frac{d^2U}{dw^2},$$

$$l = \frac{d^2U}{dydz}, \quad m = \frac{d^2U}{dzdx}, \quad n = \frac{d^2U}{dxdy}, \quad p = \frac{d^2U}{dxdw}, \quad q = \frac{d^2U}{dydw}, \quad r = \frac{d^2U}{dzdw},$$

Hesse's functional determinant will be found to be

$$l^2p^2 + m^2q^2 + n^2r^2 - 2mnqr - 2nrlp - 2lmpq + abcd + 2alqr$$
$$+ 2bmpr + 2cnpq + 2dlmn - abr^2 - bcp^2 - caq^2 - adl^2 - bdm^2 - cdn^2 = 0.$$

The name "functional determinant," however, is inconvenient both for its length and for its ambiguity; for the name determinant is usually given to the result of eliminating xyz between $\frac{dU}{dx} = 0$, $\frac{dU}{dy} = 0$, $\frac{dU}{dz} = 0$. In the Cambridge and Dublin Math. Jour., vol. vi. p. 186, &c., Mr. Sylvester calls the function we are now discussing, Hessian, after M. Hesse; and though the name sounds odd, yet it is so convenient to have a precise and distinctive name, that we use the notation HU, which the learner may read, if he pleases, the Hessian of U.

moreover, that the two curves have at that point the same tangents.

The simplest proof of this is to take the double point for our origin, and the two tangents at it for our axes; the equation of the curve is then of the form

$$xyz^{n-2} + u_3 z^{n-3} + \&c. = 0.$$

The lowest powers of x and y in the following will be in

$$\frac{d^2U}{dx^2} = \frac{d^2u_3}{dx^2} z^{n-3} + \&c. \quad \frac{d^2U}{dy^2} = \frac{d^2u_3}{dy^2} z^{n-3} + \&c.$$

$$\frac{d^2U}{dz^2} = (n-2)(n-3)\, xyz^{n-4} + \&c.$$

$$\frac{d^2U}{dydz} = (n-2)\, xz^{n-3} + \&c. \quad \frac{d^2U}{dzdx} = (n-2)\, yz^{n-3} + \&c. \quad \frac{d^2U}{dxdy} = z^{n-2} + \&c.$$

Every term in H will then contain powers of x and y above the second, except

$$\frac{d^2U}{dz^2}\left(\frac{d^2U}{dxdy}\right)^2 - 2\frac{d^2U}{dxdy}\cdot\frac{d^2U}{dydz}\cdot\frac{d^2U}{dzdx},$$

and the only terms below the third degree to which these give rise will be $-(n-1)(n-2)\, xyz^{3n-8}$. Hence the origin is also a double point on H, and xy are the tangents at it.

Now, when two curves have a common double point, and the tangents at it common, this point counts for six in the number of their intersections (p. 31). Hence, if a curve have δ double points, the number of its points of inflexion will be

$$3n(n-2) - 6\delta.$$

85. If the curve have a cusp, this point will be a triple point on H, two of the tangents at which will coincide with the tangent at the cusp. For, as before, take the origin at the point, and let $x = 0$ be the tangent at the cusp: then the equation of the curve is of the form

$$x^2 z^{n-2} + u_3 z^{n-3} + \&c. = 0\,;$$

and we have

$$\frac{d^2U}{dx^2} = 2z^{n-2} + \&c., \quad \frac{d^2U}{dy^2} = \frac{d^2u_3}{dy^2} z^{n-3} + \&c.,$$

$$\frac{d^2U}{dz^2} = (n-2)(n-3)\, x^2 z^{n-4} + \&c.,$$

$$\frac{d^2U}{dydz} = (n-3)\frac{du_3}{dy}z^{n-4} + \&c.,\quad \frac{d^2U}{dxdz} = 2(n-2)\,xz^{n-3} + \&c.,$$

$$\frac{d^2U}{dydx} = \frac{d^2u_3}{dxdy}z^{n-3} + \&c.$$

The only terms then in H, below the fourth degree in x and y, will be contained in

$$\frac{d^2U}{dy^2}\left(\frac{d^2U}{dxdz}\right)^2 - \frac{d^2U}{dx^2}\cdot\frac{d^2U}{dy^2}\cdot\frac{d^2U}{dz^2},$$

and they will be $\quad 2(n-1)(n-2)\dfrac{d^2u_3}{dy^2}x^2z^{3n-8}.$

Hence, one of the tangents at the triple point is $\dfrac{d^2u_3}{dy^2}$, but the other two coincide with x. Now when two curves have common a point, which is a double point on one, and a triple on the other, this point would count for six intersections (Note, p. 31): but if, moreover, the two tangents at the double point be also tangents at the triple point, the curves have two more consecutive points common, and therefore this point would count for eight intersections. Hence, if a curve have δ double points, and κ cusps, the number of its points of inflexion will be $3n(n-2) - 6\delta - 8\kappa$.

86. And so in like manner for a multiple point of any order k. The equation of the curve being then of the form $u_k z^{n-k} + \&c. = 0$, the lowest powers of x and y will be of the degrees

$$\text{in } \frac{d^2U}{dx^2} = k-2;\ \text{in } \frac{d^2U}{dy^2} = k-2;\ \text{in } \frac{d^2U}{dz^2} = k;$$

$$\text{in } \frac{d^2U}{dydz} = k-1;\ \text{in } \frac{d^2U}{dxdz} = k-1;\ \text{in } \frac{d^2U}{dxdy} = k-2.$$

Hence, the origin will be on the curve H a multiple point of the order $3k-4$. But, morever, all the k tangents at the multiple point of U will be also tangents at the same point on H. For, suppose that x had been a factor in u_k, it would also be a factor in the lowest term of $\dfrac{d^2U}{dydz}$, of $\dfrac{d^2U}{dy^2}$, and of $\dfrac{d^2U}{dz^2}$, and therefore also in the lowest term of H. The multiple point will therefore count among the points of intersection as

$$k(3k-4) + k = 6k\left(\frac{k-1}{2}\right).$$

That is to say, *the multiple point has exactly the same effect on the number of points of inflexion as the equivalent number of double points*. (See Art. 73).

87. Mr. Hesse has proved (Crelle's Journal, vol. xxviii.) that in the case of curves of the third degree, every point of inflexion on U is also a point of inflexion on H; or, in other words, that if we form the equation of the curve which determines the points of inflexion, then H(HU) must be of the form AU + B.HU, passing through all the points of intersection of H and U. We shall hereafter give a special proof of the theorem for the case of the third degree, but as geometers have been naturally led to suspect that the theorem is generally true (see Crelle's Journal, vol. xxxiv. p. 44), we think it well to give here an elementary investigation, which will show that the theorem is true for curves of the third degree, but not so in general. We shall take the origin at the point of inflexion, and the tangent at it for one of the axes: the form of the equation of the curve will then be

$$Axz^{n-1} + (Bx^2 + Cxy)z^{n-2} + (Dx^3 + Ex^2y + Fxy^2 + Gy^3)z^{n-3}$$
$$+ (Hx^4 + Kx^3y + Lx^2y^2 + Mxy^3 + Ny^4)z^{n-4} \text{ \&c.} = 0.$$

To see now what will be the lowest powers of x and y in H, we must examine what the lowest powers are in the second differential coefficients. Making, for shortness, $z = 1$, these are in

$$\frac{d^2U}{dx^2} = 2B + 6Dx + 2Ey + \text{\&c.}$$

$$\frac{d^2U}{dy^2} = 2Fx + 6Gy + 2Lx^2 + 6Mxy + 12Ny^2 + \text{\&c.}$$

$$\frac{d^2U}{dz^2} = (n-1)(n-2)Ax + (n-2)(n-3)(Bx^2 + Cxy) + \text{\&c.}$$

$$\frac{d^2U}{dydz} = (n-2)Cx + (n-3)(Ex^2 + 2Fxy + 3Gy^2) + \text{\&c.}$$

$$\frac{d^2U}{dxdz} = (n-1)A + (n-2)(2Bx + Cy) + \text{\&c.}$$

$$\frac{d^2U}{dxdy} = C + 2Ex + 2Fy + \text{\&c.}$$

The terms, then, of the first degree in x and y, which denote the tangent to H, will be

$$(n-1)A[\{2(n-1)AF - (n-2)C^2\}x + 6(n-1)AGy].$$

The terms of the second degree in x and y will be

$$2(n-2)(Bx+Cy)\left[\{2(n-1)AF-(n-2)C^2\}x+6(n-1)AGy\right]$$
$$+(n-1)^2A^2(2Lx^2+6Mxy+12Ny^2)+(n-2)(n-3)C^2(Bx^2+Cxy)$$
$$-2(n-1)(n-3)AC(Ex^2+2Fxy+3Gy^2).$$

Now these terms are not, in general, divisible by the equation of the tangent; but if $n=3$, all but the first group of terms vanish, and then the equation of the tangent does enter as a factor into the terms of the second degree, and therefore the origin is a point of inflexion on H.

88. We shall briefly indicate the eliminations necessary to be performed, in order to obtain the equations of the $3n(n-2)$ tangents at the points of inflexion. If, in the general equation of a tangent,

$$x\left(\frac{dU}{dx}\right)_1+y\left(\frac{dU}{dy}\right)_1+z\left(\frac{dU}{dz}\right)_1=0,$$

$x_1y_1z_1$ fulfil not only the condition $U_1=0$, but also $H=0$, it will plainly be a tangent at one of the points of intersection of U and H. If between the three equations we eliminate $x_1y_1z_1$, we shall have the equation of the system of tangents at all the points of intersection. Since xyz enter only into the first equation, and there only in the first degree, and since the other two equations are of the degrees n and $3(n-2)$ respectively in $x_1y_1z_1$, the final result will be of the degree $3n(n-2)$ in xyz, as it plainly ought to be.*

Or we might have obtained this result without having first obtained the equation $H=0$. We saw (Art. 83) that between a point of inflexion and any point on its tangent the following relations subsist:

$$U_1=0,\quad \Delta U_1=0,\quad \Delta^2U_1=0.$$

If, then, between these three equations we eliminate $x_1y_1z_1$, we shall have the locus of all the points whose first and second polars intersect on the curve; a locus which the preceding equations show will include all the tangents at points of inflexion. But every point on the curve is also a point on this locus, for we have

* For the degree of the result of elimination between several given equations, see Note on Elimination at the end of the volume.

seen (Art. 65) that the polars of every point on the curve touch the curve at that point. Now, in the result of elimination the coefficients of ΔU_1 enter in the degree $n(n-2)$, and those of $\Delta^2 U_1$ in the degree $n(n-1)$; and since the first equation contains xyz in the first, and the second contains xyz in the second degree, the final result will be of the degree $n(3n-4)$ in xyz, and it will be the equation of the $3n(n-2)$ tangents multiplied by the square of the equation of the curve.

89. Although what has been said contains the complete theory of the points of inflexion, it will be instructive to give also the way in which Mr. Cayley has solved the same problem. (Crelle's Journal, vol. xxxiv. p. 30.) The question was reduced (Art. 83) to finding the condition that the conic,

$$\Delta^2 U_1 = x^2\left(\frac{d^2U}{dx^2}\right)_1 + y^2\left(\frac{d^2U}{dy^2}\right)_1 + z^2\left(\frac{d^2U}{dz^2}\right)_1 + 2xy\left(\frac{d^2U}{dxdy}\right)_1 + 2yz\left(\frac{d^2U}{dydz}\right)_1 + 2zx\left(\frac{d^2U}{dzdx}\right)_1 = 0,$$

should have as a factor the right line

$$\Delta U_1 = x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0.$$

We obtained the condition directly by applying the criterion that the conic should break up into two right lines. It is plain, however, that such a method would not, in general, be applicable to solve the problem of finding the conditions that two equations, $U=0$, $V=0$, should have a common factor. If, however, with the two given equations we combine a third, assumed arbitrarily (for simplicity we take one of the first degree),

$$\alpha x + \beta y + \gamma z = 0,$$

then if U and V have a common factor, since *every* line must pass through a point common to U and V, the result of elimination between $U=0$, $V=0$, and $\alpha x+\beta y+\gamma z=0$, must be satisfied, independently of any particular values of $\alpha\beta\gamma$.

To satisfy this independently of $\alpha\beta\gamma$, would give rise to more conditions than one; for, in general, three conditions will be required, in order that a conic should contain a given line as a factor.

The reason that we now require only one new condition is, that we have already proved (Art. 65) that when $x_1y_1z_1$ is on the curve, ΔU_1 and $\Delta^2 U_1$ touch each other, that is, have two consecutive points common. In order, then, that ΔU_1 should be a factor in $\Delta^2 U_1$, it is only necessary that they should have a third point common. And this consideration enables us readily to write down the form of the result of elimination. For, in the general case, if $x_1y_1z_1$, $x_2y_2z_2$, were the points of intersection of Δ, Δ^2, the result of elimination between them and $\alpha x + \beta y + \gamma z$ must be (see Note on Elimination) of the form

$$H(\alpha x_1 + \beta y_1 + \gamma z_1)(\alpha x_2 + \beta y_2 + \gamma z_2) = 0;$$

H being some factor independent of $\alpha\beta\gamma$. When, therefore, $x_1y_1z_1$ is on the curve, the result must be of the form

$$H(\alpha x_1 + \beta y_1 + \gamma z_1)^2 = 0.$$

Or otherwise; we can see that an arbitrary right line, $\alpha x + \beta y + \gamma z$, will pass through the intersection of ΔU_1 and $\Delta^2 U_1$, either if it be taken so as to pass through the point $x_1y_1z_1$ ($\alpha x_1 + \beta y_1 + \gamma z_1 = 0$), or if $x_1y_1z_1$ be a point of inflexion ($H = 0$). The degree of H is what we are now concerned with.

Now, the result of elimination between the conic $\Delta^2 U$ and the two right lines ΔU_1 and $\alpha x + \beta y + \gamma z$, contains $\alpha\beta\gamma$ in the second degree, and $x_1y_1z_1$ in the $3n - 4^{th}$ degree; since $x_1y_1z_1$ enter in the $(n-1)^{st}$ degree into ΔU_1, and in the $(n-2)^{nd}$ degree into $\Delta^2 U_1$. It follows, then, that the factor H contains only $x_1y_1z_1$, and that in the degree $3(n-2)$, as was otherwise proved in Art. 83.

90. It will be useful for some future applications to give the actual formation of the function H by this method. In this and the succeeding Articles we shall use the following abbreviations:

$$\frac{dU}{dx} = L; \quad \frac{dU}{dy} = M; \quad \frac{dU}{dz} = N.$$

We shall denote the second differential coefficients by A, B, C, D, E, F, as at Art. 75; and we shall write

$$CF - E^2 = \mathfrak{A}, \quad DE - BF = \mathfrak{B}, \quad AF - D^2 = \mathfrak{C},$$
$$BE - CD = \mathfrak{D}, \quad BD - AE = \mathfrak{E}, \quad AC - B^2 = \mathfrak{F}.$$

The co-ordinates then of an arbitrary point on the tangent, found from the equations

$$Lx + My + Nz = 0, \quad \alpha x + \beta y + \gamma z = 0,$$

are $\qquad x = \gamma M - \beta N, \quad y = \alpha N - \gamma L, \quad z = \beta L - \alpha M;$

and if we substitute these co-ordinates in

$$\Delta^2 U = Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2,$$

we shall have the result of elimination between this equation and the two just written; which we have seen in the last Article must, by the help of the equation of the curve, be reducible to the form

$$H(\alpha x_1 + \beta y_1 + \gamma z_1)^2.$$

Now, for simplicity, we shall make $\alpha = 0$, $\beta = 0$, $\gamma = 1$, in the above equations, and shall prove that the result of substituting in $\Delta^2 U_1$ ($x = M$, $y = -L$, $z = 0$) is, by the help of the equation of the curve, reducible to the form $z_1^2 H$. The geometrical meaning of this simplification is, that instead of examining whether an arbitrary point on the tangent be also on $\Delta^2 U$, we examine whether the particular point in which the tangent meets the line z be on $\Delta^2 U$; and the result will show that this will be the case either if the point of contact of the tangent be on z, or if $x_1 y_1 z_1$ satisfy the condition $H = 0$. By the substitution in question, then, $\Delta^2 U$ becomes

$$\Delta = AM^2 - 2BLM + CL^2.$$

But, by the theory of homogeneous functions,

$$(n-1) L = Ax + By + Dz,$$
$$(n-1) M = Bx + Cy + Ez,$$
$$(n-1) N = Dx + Ey + Fz.$$

Hence
$$(n-1)(AM - BL) = \mathfrak{F}y - \mathfrak{E}z,$$
$$(n-1)(CL - BM) = \mathfrak{F}x - \mathfrak{D}z.$$

Therefore

$$(n-1)\{M(AM - BL) + L(CL - BM)\}$$
$$= \mathfrak{F}(Lx + My + Nz) - z(\mathfrak{D}L + \mathfrak{E}M + \mathfrak{F}N).$$

Now, if we put for L, M, N their values just given, it will be seen that the coefficients of x and y in $\mathfrak{D}L + \mathfrak{E}M + \mathfrak{F}N$ vanish identically, and that the coefficient of z gives

$$(n-1)(\mathfrak{D}L + \mathfrak{E}M + \mathfrak{F}N)$$
$$= -z(AE^2 + CD^2 + FB^2 - ACF - 2BDE) = -Hz.$$

Hence
$$(AM^2 - 2BLM + CL^2) = \frac{n}{n-1}\mathfrak{F}U + \frac{1}{(n-1)^2}Hz^2.$$

When, therefore, $x_1y_1z_1$ satisfy the equation $U = 0$, this reduces to $\dfrac{Hz^2}{(n-1)^2}$.

91. We proceed now to apply the same method to the investigation of multiple tangents of higher orders. We may first, however, remark, that when a curve has a point of *undulation* (at which the tangent meets the curve in four consecutive points), then the curve H touches the curve U at that point. For, take the origin at the point, then, as in Art. 87, the equation of the curve U will be of the form

$$Axz^{n-1} + (Bx^2 + Cxy)\,z^{n-2} - (Dx^3 + Ex^2y + Fxy^2)\,z^{n-3} + \&c. = 0;$$

and the equation of the tangent to H, found by making $G = 0$ in that given in Art. 87, is simply $x = 0$. Since then H and U intersect thus in two coincident points, we learn that *a point of undulation arises from the union of two points of inflexion.* The same thing appears from the fact, that a line meeting the curve in four consecutive points, doubly meets it in three consecutive points, viz., in 123 and in 234.

The general condition for a point of undulation is, as is readily seen from the method pursued in Art. 83, that ΔU_1, the equation of the tangent, should be a factor not only in $\Delta^2 U_1$, but also in $\Delta^3 U_1$. If the tangent meet the curve in five consecutive points ΔU_1 must also be a factor in $\Delta^4 U_1$, and so on. Now, to obtain the conditions in general that the tangent should be a factor in $\Delta^k U$, we must, as in the preceding Articles, join to the equation of the tangent $(Lx + My + Nz)$ that of an arbitrary right line $(\alpha x + \beta y + \gamma z)$, eliminate between these two and $\Delta^k U$, and find the condition that the result should be satisfied independently of $\alpha\beta\gamma$. Or, what is the same thing, we should substitute in $\Delta^k U_1$

$$x = \gamma M - \beta N,\ \ y = \alpha N - \gamma L,\ \ z = \beta L - \alpha M.$$

The result of elimination must, in general, be of the form

$$(\alpha x_1 + \beta y_1 + \gamma z_1)\ (\alpha x_2 + \beta y_2 + \gamma z_2)\ (\alpha x_3 + \beta y_3 + \gamma z_3)\ (\&c.) = 0,$$

where $x_1y_1z_1$ &c. are the co-ordinates of the points of intersection of ΔU_1 and $\Delta^k U_1 = 0$. But when $x_1y_1z_1$ is on the curve, since every polar curve is touched by the tangent, two of these points coincide with $x_1y_1z_1$, and the result must reduce to the form

$$(\alpha x_1 + \beta y_1 + \gamma z_1)^2\,Q = 0.$$

Now Mr. Hesse has shown directly (Crelle's Journal, vol. xxxvi. p. 143) that the result of this substitution is always

$$\Delta^k = P_k U + Q_k(\alpha x_{,} + \beta y_{,} + \gamma z_{,})^2,$$

which reduces when $x_{,}y_{,}z_{,}$ is on the curve to $Q_k(\alpha x_{,} + \beta y_{,} + \gamma z_{,})^2$. His method of proof is to show, that if this be true for two consecutive Δ^{k-1}, Δ^k, it must be also true for Δ^{k+1}; and he has given a rule for forming the result of substitution in Δ^{k+1}, in terms of Δ^k and Δ^{k-1}. We refer to Mr. Hesse's paper any reader desirous of seeing how the general case is to be treated; but since, in all the applications which we shall have to make, it is sufficient to take $\alpha = 0$, $\beta = 0$, $\gamma = 1$, we shall simplify Mr. Hesse's analysis, and content ourselves with examining the result of substituting in $\Delta^k U$, $x = M$, $y = -L$, $z = 0$.

92. By this substitution, then, it is easy to see that

$$\Delta^{k-1} = \frac{d^{k-1}U}{dx^{k-1}} M^{k-1} - (k-1)\frac{d^{k-1}U}{dx^{k-2}dy} M^{k-2}L + \frac{(k-1)(k-2)}{1.2}\frac{d^{k-1}U}{dx^{k-3}dy^2} M^{k-3}L^2 - \&c.$$

$$\Delta^{k} = \frac{d^{k}U}{dx^{k}} M^{k} - k\frac{d^{k}U}{dx^{k-1}dy} M^{k-1}L + \frac{k(k-1)}{1.2}\frac{d^{k}U}{dx^{k-2}dy^2} M^{k-2}L^2 - \&c.$$

$$\Delta^{k+1} = \frac{d^{k+1}U}{dx^{k+1}} M^{k+1} - (k+1)\frac{d^{k+1}U}{dx^{k}dy} M^{k}L + \frac{(k+1)k}{1.2}\frac{d^{k+1}U}{dx^{k-1}dy^2} M^{k-1}L^2 - \&c.$$

It is obvious, then, that $\Delta^{k+1} = M\left[\frac{d\Delta^k}{dx}\right] - L\left[\frac{d\Delta^k}{dy}\right]$, where, by $\left[\frac{d\Delta^k}{dx}\right]$ we mean the differential, on the supposition that L and M are constant. But the complete differential is found by adding to the differential on this supposition,

$$\frac{d\Delta^k}{dL}\cdot\frac{dL}{dx} + \frac{d\Delta^k}{dM}\cdot\frac{dM}{dx};$$

or

$$\left[\frac{d\Delta^k}{dx}\right] = \frac{d\Delta^k}{dx} - A\frac{d\Delta^k}{dL} - B\frac{d\Delta^k}{dM}; \quad \left[\frac{d\Delta^k}{dy}\right] = \frac{d\Delta^k}{dy} - B\frac{d\Delta^k}{dL} - C\frac{d\Delta^k}{dM}.$$

Hence,

$$\Delta^{k+1} = \mathrm{M}\frac{d\Delta^k}{dx} - \mathrm{L}\frac{d\Delta^k}{dy} + (\mathrm{BL} - \mathrm{AM})\frac{d\Delta^k}{d\mathrm{L}} + (\mathrm{CL} - \mathrm{BM})\frac{d\Delta^k}{d\mathrm{M}}.$$

Again, it is obvious that

$$\frac{d\Delta^k}{d\mathrm{M}} = k\left[\frac{d\Delta^{k-1}}{dx}\right]; \quad \frac{d\Delta^k}{d\mathrm{L}} = -k\left[\frac{d\Delta^{k-1}}{dy}\right];$$

and

$$\left[\frac{d\Delta^{k-1}}{dx}\right] = \frac{d\Delta^{k-1}}{dx} - \mathrm{A}\frac{d\Delta^{k-1}}{d\mathrm{L}} - \mathrm{B}\frac{d\Delta^{k-1}}{d\mathrm{M}};$$

$$\left[\frac{d\Delta^{k-1}}{dy}\right] = \frac{d\Delta^{k-1}}{dy} - \mathrm{B}\frac{d\Delta^{k-1}}{d\mathrm{L}} - \mathrm{C}\frac{d\Delta^{k-1}}{d\mathrm{M}}.$$

Substituting these values, we have

$$\Delta^{k+1} = \left(\mathrm{M}\frac{d\Delta^k}{dx} - \mathrm{L}\frac{d\Delta^k}{dy}\right) + k(\mathrm{CL} - \mathrm{BM})\frac{d\Delta^{k-1}}{dx}$$
$$+ k(\mathrm{AM} - \mathrm{BL})\frac{d\Delta^{k-1}}{dy} - k\mathfrak{F}\left(\mathrm{L}\frac{d\Delta^{k-1}}{d\mathrm{L}} + \mathrm{M}\frac{d\Delta^{k-1}}{d\mathrm{M}}\right);$$

or since Δ^{k-1} is a homogeneous function of L and M of the degree $k-1$,

$$\Delta^{k+1} = \left(\mathrm{M}\frac{d\Delta^k}{dx} - \mathrm{L}\frac{d\Delta^k}{dy}\right) + k(\mathrm{CL} - \mathrm{BM})\frac{d\Delta^{k-1}}{dx}$$
$$+ k(\mathrm{AM} - \mathrm{BL})\frac{d\Delta^{k-1}}{dy} - k(k-1)\mathfrak{F}\Delta^{k-1}.$$

This equation may be thrown into a somewhat different form by putting in for (CL – BM) and (AM – BL) their values given in Art. 90, when we get

$$\Delta^{k+1} = \left(\mathrm{M}\frac{d\Delta^k}{dx} - \mathrm{L}\frac{d\Delta^k}{dy}\right) + \frac{k}{n-1}\mathfrak{F}\left(x\frac{d\Delta^{k-1}}{dx} + y\frac{d\Delta^{k-1}}{dy} + z\frac{d\Delta^{k-1}}{dz}\right)$$
$$- \frac{kz}{n-1}\left(\mathfrak{D}\frac{d\Delta^{k-1}}{dx} + \mathfrak{E}\frac{d\Delta^{k-1}}{dy} + \mathfrak{F}\frac{d\Delta^{k-1}}{dz}\right) - k(k-1)\mathfrak{F}\Delta^{k-1}.$$

But since Δ^{k-1} is a homogeneous function of xyz of the degree $nk - 2k + 2$, and since $\frac{nk - 2k + 2}{n-1} - (k-1) = \frac{n-k+1}{n-1}$, we have

$$\Delta^{k+1} = \left(\mathrm{M}\frac{d\Delta^k}{dx} - \mathrm{L}\frac{d\Delta^k}{dy}\right) + \frac{k(n-k+1)}{n-1}\mathfrak{F}\Delta^{k-1}$$
$$- \frac{kz}{n-1}\left(\mathfrak{D}\frac{d\Delta^{k-1}}{dx} + \mathfrak{E}\frac{d\Delta^{k-1}}{dy} + \mathfrak{F}\frac{d\Delta^{k-1}}{dz}\right).$$

From this equation we can readily prove that if

$$\Delta^{k-1} = P_{k-1}U + z^2 Q_{k-1} \text{ and } \Delta^k = P_k U + z^2 Q_k,$$

then Δ^{k+1} will also be of the form $P_{k+1}U + z^2 Q_{k+1}$. In fact, substituting these values for Δ^{k-1}, Δ^k, and observing that

$$\frac{d\Delta^{k-1}}{dx} = \frac{dP_{k-1}}{dx} U + P_{k-1}L + z^2 \frac{dQ_{k-1}}{dx},$$

$$\frac{d\Delta^{k-1}}{dy} = \frac{dP_{k-1}}{dy} U + P_{k-1}M + z^2 \frac{dQ_{k-1}}{dy},$$

$$\frac{d\Delta^{k-1}}{dz} = \frac{dP_{k-1}}{dz} U + P_{k-1}N + z^2 \frac{dQ_{k-1}}{dz} + 2zQ_{k-1},$$

and remembering that we had occasion in Art. 90 to see that

$$(n-1)(\mathfrak{D}L + \mathfrak{E}M + \mathfrak{F}N) = -Hz,$$

we find

$$P_{k+1} = \left(M\frac{dP_k}{dx} - L\frac{dP_k}{dy}\right) + \frac{k(n-k+1)}{n-1}\mathfrak{F}P_{k-1} - \frac{kz}{n-1}\left(\mathfrak{D}\frac{dP_{k-1}}{dx} + \mathfrak{E}\frac{dP_{k-1}}{dy} + \mathfrak{F}\frac{dP_{k-1}}{dz}\right),$$

$$Q_{k+1} = \left(M\frac{dQ_k}{dx} - L\frac{dQ_k}{dy}\right) + \frac{k(n-k-1)}{(n-1)}\mathfrak{F}Q_{k-1} - \frac{kz}{n-1}\left(\mathfrak{D}\frac{dQ_{k-1}}{dx} + \mathfrak{E}\frac{dQ_{k-1}}{dy} + \mathfrak{F}\frac{dQ_{k-1}}{dy}\right) + \frac{kH}{(n-1)^2}P_{k-1}.$$

93. From these formulæ we can form a table of the values of P_3, Q_3, P_4, Q_4, &c. For it is obvious that $P_1 = 0$, $Q_1 = 0$; and we proved in Art. 90 that $P_2 = \frac{n}{n-1}\mathfrak{F}$, and $Q_2 = \frac{1}{(n-1)^2}H$. Hence

$$Q_3 = \frac{1}{(n-1)^2}\left(M\frac{dH}{dx} - L\frac{dH}{dy}\right).$$

Again,

$$Q_4 = \left(M\frac{dQ_3}{dx} - L\frac{dQ_3}{dy}\right) + \frac{3(n-4)}{(n-1)^3}\mathfrak{F}H + \frac{3n}{(n-1)^3}\mathfrak{F}H - \frac{3z}{(n-1)^3}\left(\mathfrak{D}\frac{dH}{dx} + \mathfrak{E}\frac{dH}{dy} + \mathfrak{F}\frac{dH}{dz}\right).$$

But

$$\left(M\frac{dQ_3}{dx} - L\frac{dQ_3}{dy}\right) = \frac{1}{(n-1)^2}\left(M^2\frac{d^2H}{dx^2} - 2ML\frac{d^2H}{dxdy} + L^2\frac{d^2H}{dy^2}\right) + \frac{1}{(n-1)^2}(BM - CL)\frac{dH}{dx} - \frac{1}{(n-1)^2}(AM - BL)\frac{dH}{dy}.$$

And from the values of BM − CL, AM − BL, already used, we have

$$(n-1)\left\{(\mathrm{BM}-\mathrm{CL})\frac{d\mathrm{H}}{dx}-(\mathrm{AM}-\mathrm{BL})\frac{d\mathrm{H}}{dy}\right\}$$

$$=-\mathfrak{F}\left(x\frac{d\mathrm{H}}{dx}+y\frac{d\mathrm{H}}{dy}+z\frac{d\mathrm{H}}{dz}\right)+z\left(\mathfrak{D}\frac{d\mathrm{H}}{dx}+\mathfrak{E}\frac{d\mathrm{H}}{dy}+\mathfrak{F}\frac{d\mathrm{H}}{dz}\right).$$

Hence

$$\mathrm{Q}_4=\frac{1}{(n-1)^2}\left(\mathrm{M}^2\frac{d^2\mathrm{H}}{dx^2}-2\mathrm{ML}\frac{d^2\mathrm{H}}{dxdy}+\mathrm{L}^2\frac{d^2\mathrm{H}}{dy^2}\right)$$

$$+\frac{3(n-2)}{(n-1)^3}\mathfrak{F}\mathrm{H}-\frac{2z}{(n-1)^3}\left(\mathfrak{D}\frac{d\mathrm{H}}{dx}+\mathfrak{E}\frac{d\mathrm{H}}{dy}+\mathfrak{F}\frac{d\mathrm{H}}{dz}\right).$$

And so in like manner we can form the rest.

94. It has been proved that the result of elimination between $\mathrm{L}x+\mathrm{M}y+\mathrm{N}z$, $\alpha x+\beta y+\gamma z$, and $\Delta^3\mathrm{U}_1$, is, in general, of the form

$$(\alpha x_1+\beta y_1+\gamma z_1)^2\ (\alpha x_3+\beta y_3+\gamma z_3).$$

Now by taking $\alpha=0$, $\beta=0$, we have just proved that z_3 must be proportional to $\mathrm{Q}_3=\mathrm{M}\frac{d\mathrm{H}}{dx}-\mathrm{L}\frac{d\mathrm{H}}{dy}$; and we can in like manner see the values of x_3, y_3; so that the result of elimination must be, in general,

$$(\alpha x_1+\beta y_1+\gamma z_1)^2\left\{\alpha\left(\mathrm{N}\frac{d\mathrm{H}}{dy}-\mathrm{M}\frac{d\mathrm{H}}{dz}\right)+\beta\left(\mathrm{L}\frac{d\mathrm{H}}{dz}-\mathrm{N}\frac{d\mathrm{H}}{dx}\right)\right.$$

$$\left.+\gamma\left(\mathrm{M}\frac{d\mathrm{H}}{dx}-\mathrm{L}\frac{d\mathrm{H}}{dy}\right)\right\},$$

a value first obtained by Mr. Cayley (Crelle's Journal, vol. xxxiv. p. 43).

When the curve is of the third degree, $\Delta^3\mathrm{U}_1$ is no other than the equation of the curve itself, and the result just given leads (as Mr. Hesse has pointed out) to the following consequence: "If any line, $\alpha x+\beta y+\gamma z$, meet a curve of the third degree, and from each of the points where it meets it four tangents (Art. 79) be drawn to the curve, the twelve points of contact lie on the curve of the fourth degree,

$$\alpha\left(\mathrm{N}\frac{d\mathrm{H}}{dy}-\mathrm{M}\frac{d\mathrm{H}}{dz}\right)+\beta\left(\mathrm{L}\frac{d\mathrm{H}}{dz}-\mathrm{N}\frac{d\mathrm{H}}{dx}\right)+\gamma\left(\mathrm{M}\frac{d\mathrm{H}}{dx}-\mathrm{L}\frac{d\mathrm{H}}{dy}\right)=0."$$

For this condition must, as we have seen, be fulfilled by any

point of the curve, such that the point where its tangent meets $\alpha x + \beta y + \gamma z$ lies on the curve.

If we seek the co-ordinates of the point where any tangent to a curve of the third degree meets the curve again, we see that these must be proportional to

$$N\frac{dH}{dy} - M\frac{dH}{dz},\ L\frac{dH}{dz} - N\frac{dH}{dx},\ M\frac{dH}{dx} - L\frac{dH}{dy}.$$

Or these co-ordinates may be put into a form which will show that where the point of contact satisfies the condition $H = 0$, the third point of contact will coincide with $x_1y_1z_1$, for we have

$$3H = x_1\frac{dH}{dx} + y_1\frac{dH}{dy} + z_1\frac{dH}{dz};\ Lx_1 + My_1 + Nz_1 = 0.$$

Hence,

$$y_1x_3 - x_1y_3 = 3NH;\ z_1y_3 - y_1z_3 = 3LH;\ x_1z_3 - z_1x_3 = 3MH;$$

equations which the supposition $H = 0$ reduces to

$$\frac{x_3}{x_1} = \frac{y_3}{y_1} = \frac{z_3}{z_1}.$$

95. We shall now proceed to the problem of finding the double tangents whose points of contact do not coincide. Let the line joining the points $x_1y_1z_1$, xyz touch the curve at the former point. Then the equation Z which determines the other points where this line meets the curve, becomes

$$\lambda^{n-2}\Delta^2U_1 + \frac{\lambda^{n-3}\mu}{3}\Delta^3U_1 + \&c. + \mu^{n-2}U = 0.$$

If then this line touch the curve again, this equation must have equal roots. Let the condition that this should be the case (obtained by eliminating between the differentials of this equation with respect to λ, μ) be expressed as $\phi = 0$; then since this equation must be satisfied for every point on the tangent, it must contain ΔU_1 as a factor. In order, then, to determine when a double tangent is possible, we have only to investigate the conditions that ΔU_1 should be a factor in ϕ.

Or the same thing may appear thus. We showed (Art. 80) that two of the tangents which can be drawn to a curve from a point on it coincide with the tangent at that point, and that the equation of the rest is of the form

$$k\Delta U_1 + (\Delta^2U_1)^2\phi = 0.$$

But if the tangent at the point be a double tangent, a third tangent will coincide with the tangent at the point, and therefore the preceding equation must be divisible by ΔU_1. This will happen either when ΔU_1 is a factor in $\Delta^2 U_1$ or in ϕ. The first is the case of a point of inflexion, the second of a double tangent.

96. In order to ascertain whether ϕ contains ΔU_1 as a factor, we follow the method of Art. 89; we combine with both the equation of an arbitrary line, and eliminate xyz. Now (see Note on Elimination) the condition that any equation $Ax^n + \ldots + Z$ should have equal roots, is of the degree $2(n-1)$ in the coefficients of that equation, and one of the terms is $(AZ)^{n-1}$. Hence one of the terms in ϕ will be $(U.\Delta^2 U_1)^{n-3}$; ϕ is therefore of the degree $(n+2)(n-3)$ in xyz, of the degree $(n-2)(n-3)$ in $x_1y_1z_1$, and of the degree $2(n-3)$ in the coefficients of the original equation.

The result of elimination, then, between ϕ, ΔU_1, and $ax + \beta y + \gamma z$ will be of the degree $(n+2)(n-3)$ in $a\beta\gamma$; of the degree $(n^2+2n-4)(n-3)$ in $x_1y_1z_1$, and of the degree $(n+4)(n-3)$ in the coefficients of the original equation.

Now, as before, the result of elimination must be of the form

$$\Pi(a\xi_1 + \beta\eta_1 + \gamma\zeta_1)(a\xi_2 + \beta\eta_2 + \gamma\xi_2)(\&c.) = 0,$$

where ξ, η, ζ are the co-ordinates of the intersections of ΔU_1, ϕ. But we shall now show that all these coincide with $x_1y_1z_1$. For the system of $n^2 - n - 2$ tangents through $x_1y_1z_1$, whose equation is $k\Delta U_1 + (\Delta^2 U_1)^2\phi = 0$, can be met by ΔU_1 in no point but $x_1y_1z_1$; make, then, $\Delta U_1 = 0$ in the above equation, and it is plain that it can meet neither $\Delta^2 U_1$ nor ϕ in any other point than $x_1y_1z_1$. The result of elimination written above must then, when $x_1y_1z_1$ is on the curve, reduce to

$$\Pi(ax_1 + \beta y_1 + \gamma z_1)^{(n+2)(n-3)} = 0,$$

Π then will not contain $a\beta\gamma$, will be of the degree $(n+3)(n-2)(n-3)$ in $x_1y_1z_1$, and of the degree $(n+4)(n-3)$ in the coefficients of the original equation. If, therefore, $x_1y_1z_1$ fulfil the condition $\Pi = 0$, any arbitrary line meets ΔU and ϕ in the same point; ΔU must therefore be a factor in ϕ, and therefore a double tangent to the curve. There will be $n(n+3)(n-2)(n-3)$ points of intersection of Π and U; and since two of these lie on

each double tangent, there will be half this number of double tangents.

97. Nothing more is necessary if it be only required to ascertain the number of double tangents which a curve of the n^{th} degree can possess, and to perceive the degree of the curve Π which passes through their points of contact. If it be required, however, actually to obtain the equation of this curve, we must form the condition that the following equation should have equal roots,

$$(\Delta^2 U_1)\lambda^{n-2} + (\Delta^3 U_1)\frac{\lambda^{n-3}\mu}{3} + (\Delta^4 U_1)\frac{\lambda^{n-4}\mu^2}{3.4} + \&c. = 0,$$

and then substitute

$$x = \gamma M - \beta N, \quad y = \alpha N - \gamma L, \quad z = \beta L - \alpha M,$$

when the result will be found divisible by

$$(\alpha x_1 + \beta y_1 + \gamma z_1)^{(n+2)\ (n-3)},$$

and will give the required function Π. It will be, however, more simple to make this substitution *before* forming the condition that the equation Z should have equal roots. For we have seen (Art. 91) that every term of the equation will then contain the factor $(\alpha x_1 + \beta y_1 + \gamma z_1)^2$; and since the condition that the equation should have equal roots contains these terms in the degree $2(n-3)$, we have the equation at once divisible by $(\alpha x_1 + \beta y_1 + \gamma z_1)^{4(n-3)}$, and it remains to show that it is still further divisible by

$$(\alpha x_1 + \beta y_1 + \gamma z_1)^{(n-2)\ (n-3)}.$$

In practice it will suffice to take $\alpha = 0$, $\beta = 0$, $\gamma = 1$, and accordingly to substitute $x = M$; $y = -L$; $z = 0$. The equation Z then becomes

$$Q_2\lambda^{n-2} + Q_3\frac{\lambda^{n-3}\mu}{3} + Q_4\frac{\lambda^{n-4}\mu^2}{3.4} + \&c. = 0,$$

where Q_2, Q_3, &c., have the values found in Arts. 92, 93. The condition that this equation should have equal roots will be of the degree $(n+4)\ (n-2)\ (n-3)$ in x, y, z, since Q_2, Q_3, &c., are of the degrees $3(n-2)$, $4(n-2)\ldots$, $(n+1)\ (n-2)$. And in this condition the terms above the degree $(n+3)\ (n-2)\ (n-3)$ in x and y will vanish identically, the equation will become divisible by $z^{(n-2)\ (n-3)}$, and will give the equation Π. This has not yet been verified in general, nor has any rule been given for writing down

the equation Π; but Mr. Hesse has very lately* given the formation of the equation Π in the case where $n = 4$, and the extension of his method to curves of higher orders seems a work rather of labour than of difficulty.

98. In the case, then, when $n = 4$, the equation Z becomes

$$\lambda^2 Q_2 + \frac{\lambda\mu}{3} Q_3 + \frac{\mu^2}{3.4} Q_4 = 0.$$

The condition that this equation should have equal roots is

$$Q_3{}^2 = 3Q_2 Q_4,$$

an equation of the 16th degree in x and y, which is to be reduced to the 14th by the help of the equation of the curve. By Art. 93 we have seen

$$9Q_2 = \text{H}\,; \quad 9Q_3 = \text{M}\frac{d\text{H}}{dx} - \text{L}\frac{d\text{H}}{dy}\,;$$

$$9Q_4 = \left(\text{M}^2\frac{d^2\text{H}}{dx^2} - 2\text{ML}\frac{d^2\text{H}}{dxdy} + \text{L}^2\frac{d^2\text{H}}{dy^2}\right) + 2\mathfrak{F}\text{H} - \frac{2z}{3}\left(\mathfrak{D}\frac{d\text{H}}{dx} + \mathfrak{E}\frac{d\text{H}}{dy} + \mathfrak{F}\frac{d\text{H}}{dz}\right).$$

Now, if we square the equation $3\text{M} = \text{B}x + \text{C}y + \text{E}z$, and subtract

$$\text{C}\,(\text{A}x^2 + 2\text{B}xy + \text{C}y^2 + 2\text{D}xz + 2\text{E}yz + \text{F}z^2),$$

which by the equation of the curve is identically $= 0$, we have

$$9\text{M}^2 = -\,\mathfrak{F}x^2 + 2\mathfrak{D}xz - \mathfrak{A}z^2.$$

Similarly,

$$9\text{L}^2 = -\,\mathfrak{F}y^2 + 2\mathfrak{E}yz - \mathfrak{C}z^2,$$

$$9\text{LM} = -\,\mathfrak{F}xy - \mathfrak{E}xz - \mathfrak{D}yz + \mathfrak{B}z^2.$$

Hence,

$$9\left(\text{M}\frac{d\text{H}}{dx} - \text{L}\frac{d\text{H}}{dy}\right)^2 = 2z\left(x\frac{d\text{H}}{dx} + y\frac{d\text{H}}{dy}\right)\left(\mathfrak{D}\frac{d\text{H}}{dx} + \mathfrak{E}\frac{d\text{H}}{dy}\right) - \mathfrak{F}\left(x\frac{d\text{H}}{dx} + y\frac{d\text{H}}{dy}\right)^2 - z^2\left(\mathfrak{A}\frac{d\text{H}^2}{dx^2} + 2\,\mathfrak{B}\frac{d\text{H}}{dx}\frac{d\text{H}}{dy} + \mathfrak{C}\frac{d\text{H}^2}{dy^2}\right).$$

But since H is a homogeneous function of the sixth degree,

$$x\frac{d\text{H}}{dx} + y\frac{d\text{H}}{dy} = 6\text{H} - z\frac{d\text{H}}{dz}.$$

* In fact Mr. Hesse's paper, containing the solution of this problem (Crelle, vol. xli. p. 285) has only reached me while this Chapter is going through the Press.

Substituting this value, we find

$$(9Q_3)^2 = -4\mathfrak{F}H^2 + \tfrac{4}{3}zH\left(\mathfrak{D}\frac{dH}{dx} + \mathfrak{E}\frac{dH}{dy} + \mathfrak{F}\frac{dH}{dz}\right) -$$

$$\tfrac{1}{9}z^2\left(\mathfrak{A}\frac{dH^2}{dx^2} + 2\mathfrak{B}\frac{dH}{dx}\frac{dH}{dy} + \mathfrak{C}\frac{dH^2}{dy^2} + 2\mathfrak{D}\frac{dH}{dx}\frac{dH}{dz} + 2\mathfrak{E}\frac{dH}{dy}\frac{dH}{dz} + \mathfrak{F}\frac{dH^2}{dz^2}\right).$$

In a precisely similar manner it is proved that

$$9\left(M\frac{d}{dx} - L\frac{d}{dy}\right)^2 H = -\mathfrak{F}\left(x\frac{d}{dx} + y\frac{d}{dy}\right)^2 H$$

$$+ 2z\left(x\frac{d}{dx} + y\frac{d}{dy}\right)\left(\mathfrak{D}\frac{d}{dx} + \mathfrak{E}\frac{d}{dy}\right)H$$

$$- z^2\left(\mathfrak{A}\frac{d^2H}{dx^2} + 2\mathfrak{B}\frac{d^2H}{dxdy} + \mathfrak{C}\frac{d^2H}{dy^2}\right);$$

and since

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right)^2 H = 30H,$$

we have

$$\left(M\frac{d}{dx} - L\frac{d}{dy}\right)^2 H = -\tfrac{10}{3}\mathfrak{F}H + \tfrac{10}{9}z\left(\mathfrak{D}\frac{dH}{dx} + \mathfrak{E}\frac{dH}{dy} + \mathfrak{F}\frac{dH}{dz}\right)$$

$$-\tfrac{1}{9}z^2\left(\mathfrak{A}\frac{d^2H}{dx^2} + 2\mathfrak{B}\frac{d^2H}{dxdy} + \mathfrak{C}\frac{d^2H}{dy^2} + 2\mathfrak{D}\frac{d^2H}{dxdz} + 2\mathfrak{E}\frac{d^2H}{dydz} + \mathfrak{F}\frac{d^2H}{dz^2}\right).$$

Hence

$$9Q_4 = -\tfrac{4}{3}\mathfrak{F}H + \tfrac{4}{9}z\left(\mathfrak{D}\frac{dH}{dx} + \mathfrak{E}\frac{dH}{dy} + \mathfrak{F}\frac{dH}{dz}\right)$$

$$-\tfrac{1}{9}z^2\left(\mathfrak{A}\frac{d^2H}{dx^2} + 2\mathfrak{B}\frac{d^2H}{dxdy} + \mathfrak{C}\frac{d^2H}{dy^2} + 2\mathfrak{D}\frac{d^2H}{dxdz} + 2\mathfrak{E}\frac{d^2H}{dydz} + \mathfrak{F}\frac{d^2H}{dz^2}\right).$$

Substituting these values for Q_2, Q_3, Q_4, in the equation

$$Q_3{}^2 = 3Q_2Q_4,$$

it becomes divisible by z^2, and is

$$\left(\mathfrak{A}\frac{dH^2}{dx^2} + 2\mathfrak{B}\frac{dH^2}{dxdy} + \mathfrak{C}\frac{dH^2}{dy^2} + 2\mathfrak{D}\frac{dH^2}{dxdz} + 2\mathfrak{E}\frac{dH^2}{dydz} + \mathfrak{F}\frac{dH^2}{dz^2}\right)$$

$$= 3H\left(\mathfrak{A}\frac{d^2H}{dx^2} + 2\mathfrak{B}\frac{d^2H}{dxdy} + \mathfrak{C}\frac{d^2H}{dy^2} + 2\mathfrak{D}\frac{d^2H}{dxdz} + 2\mathfrak{E}\frac{d^2H}{dydz} + \mathfrak{F}\frac{d^2H}{dz^2}\right).$$

This is the equation of a curve of the 14th degree, which passes through the fifty-six points of contact of double tangents to a curve of the fourth degree. It would be interesting to examine whether, perhaps, it might not be obtained more easily by geo-

metrical considerations. The left hand side, for instance, is the condition that the polar line of a point with regard to H should touch the polar conic of the same point with regard to U.*

99. It is easy to see that the number of double tangents will be reduced if the curve have double points. We have shown (Art. 30) that every line through a double point must be considered as a tangent; consequently, every such line which touches the curve elsewhere must be considered as a *double* tangent, and will give rise to points on the locus Π which do not belong to double tangents, properly so called. For instance, if the curve have one double point, there can be drawn through it (Art. 81) $n^2 - n - 6$ tangents, each one of which will count for two double tangents. If it had more double points than one, the lines joining each pair of double points would count among the double tangents. So, in like manner, if the curve had cusps or higher multiple points. We do not dwell on this direct method of obtaining the effect of multiple points on the number of multiple tangents, because the same results can be derived more simply from the theory of reciprocal curves.

SECT. V.—RECIPROCAL CURVES.

100. We have seen already (p. 1) that the degree of the reciprocal curve is always the same as the class of the given curve, and *vice versâ*. It is evident, also, that to a double point on either curve will correspond a double tangent on the other; that to a stationary point on one curve corresponds a stationary tangent on the other: and, in general, that to a multiple point of the k^{th} order corresponds a multiple tangent of the same order; that the k points of contact of the multiple tangent correspond to the k tangents at the multiple point; and that if two or more of these

* The number of double tangents to a curve of the n^{th} degree was first indirectly obtained by M. Plücker, by the help of the theory of reciprocal curves; Mr. Cayley first showed (Crelle, vol. xxxiv. p. 30) how the question was capable of being solved directly. A paper on the same subject, by Mr. Jacobi (Crelle, vol. xl. p. 237), appears to me not to have advanced the subject beyond the point where it had been left by Mr. Cayley. Finally, Mr. Hesse, in two papers (Crelle, vol. xxxvi. p. 143, and vol. xli. p. 285), by actually performing the operations indicated by Mr. Cayley, completely proved the practicability of his method.

last coincide with each other, so will the corresponding points of contact.

Let us now denote the degree of a curve by m,

,, ,, its class by n,

The number of its double points by δ,

,, ,, double tangents by τ,

,, ,, stationary points by κ,

,, ,, stationary tangents by ι,

and the corresponding numbers for the reciprocal curve are found by interchanging m and n, δ and τ, ι and κ.

We have already (see p. 63) obtained the formula

$$(1) \quad n = m^2 - m - 2\delta - 3\kappa.$$

Hence from the reciprocal curve we must have also

$$(2) \quad m = n^2 - n - 2\tau - 3\iota.$$

Again, we have obtained the formula (p. 74)

$$(3) \quad \iota = 3m^2 - 6m - 6\delta - 8\kappa.$$

Hence, by the reciprocal curve, we must have also

$$(4) \quad \kappa = 3n^2 - 6n - 6\tau - 8\iota.$$

These, however, are only equivalent to three independent equations, for, multiply the first equation by 3, and subtract from it the third, and we have

$$(5) \quad \kappa - \iota = 3(m - n),$$

an equation which may be obtained in like manner from (2) and (4).

By these equations we are able, being given any three of the numbers m, n, ι, κ, δ, τ, to determine the rest. If, for example, as we have expressed ι, n, in terms of m, δ, κ, it were also required to express τ, we have, by solving for 2τ in (2), and putting in the values of n and ι,

$$(6) \quad 2\tau = m(m-2)(m^2-9) - 2(2\delta+3\kappa)(m^2-m-6) + 4\delta(\delta-1) \\ + 9\kappa(\kappa-1) + 6\delta\kappa,$$

a result coinciding with what might have been obtained by the method of Art. 98.

A formula more convenient in practice is found by taking the difference of (1) and (2), viz.,

$$(7) \quad 2(\delta - \tau) = (m - n)(m + n - 9).$$

Thus, for example, given

$$m = 6,\ \delta = 4,\ \kappa = 6\,;$$

by formula (1), $\qquad n = 4 \therefore m - n = 2.$

Hence (5), $\qquad \kappa - \iota = 6 \therefore \iota = 0.$

$$m + n - 9 = 1 \therefore \delta - \tau = 1 \therefore \tau = 3.$$

101. We shall subject to another test the consistence of the theory of reciprocal curves just obtained. Since when a curve is given its reciprocal is determined, it is evident that the same number of conditions must suffice to determine each. Now, to be given that a curve has δ double points, is equivalent to δ conditions. Thus, for example, a curve of the second degree is determined by five conditions, but a curve of the second degree having one double point (that is, a system of two right lines) is determined by four conditions; by two points, for instance, on each of the right lines. So again, to be given that a curve has a cusp is equivalent to two conditions. Hence (and Art. 20) a curve of the m^{th} degree, with δ double points and κ cusps, is determined by $\frac{m(m+3)}{2} - \delta - 2\kappa$ conditions, and its reciprocal by $\frac{n(n+3)}{2} - \tau - 2\iota$ conditions. And by the help of the values obtained in the last Article for $\delta - \tau$, $\iota - \kappa$, it is easy to see that these two numbers are equal.

We shall give in the next section the general method of obtaining the equation of the reciprocal curve.

SECT. VI.—ENVELOPES.

102. We have already shown (*Conics*, p. 239) how to find the curve touched by a moveable line whose equation involves a variable parameter in the second degree; we proceed now to give an account of the manner in which the problem of envelopes is generally to be treated. It is plain, from Chapter I., that this section may also be considered as giving an account of the manner in which the problem of loci is to be treated in systems of tangential co-ordinates.

Let the equation of the moveable line $T = 0$ contain any va-

riable parameter t; the point of contact of the line with its envelope is plainly the point where the line intersects the consecutive line of the system: but, by Taylor's theorem, any line of the system, corresponding to $t+h$, may be written

$$T + \frac{dT}{dt}\frac{h}{1} + \frac{d^2T}{dt^2}\frac{h^2}{1.2} + \&c. = 0.$$

When, then, h is infinitely small, this equation may be considered as reducing itself to its two first terms; and the intersection of any line of the system with the consecutive one is determined by the two equations,

$$T = 0, \quad \frac{dT}{dt} = 0.$$

These two equations give the co-ordinates of the point of contact of the line answering to any particular value of t; but if between them we eliminate t, and obtain a result $U = 0$, this will be the locus of all the points of contact, that is to say, the curve touched by the moveable line.

103. If we seek the condition that three consecutive lines of the system should intersect in a point, we may see, precisely as in the last Article, that this will take place when it is possible to satisfy simultaneously the three equations

$$T = 0, \quad \frac{dT}{dt} = 0, \quad \frac{d^2T}{dt^2} = 0.$$

If we eliminate t between any two pair of these equations, we shall obtain two equations in x and y, which are sufficient to give the co-ordinates of a determinate number of points, each of which is the intersection of three consecutive lines of the system.

Since a cusp is a point at which three consecutive tangents intersect, we see that the curve U will, in general, have a determinate number of cusps. The cusps may otherwise be determined by eliminating x and y between the three equations

$$T = 0, \quad \frac{dT}{dt} = 0, \quad \frac{d^2T}{dt^2} = 0,$$

there remains then an equation in t only, which we may solve for t, and substituting any of the roots in the equation $T = 0$, we shall

have the equation of one of the cuspidal tangents. The two equations $T = 0$, $\frac{dT}{dt} = 0$, for the same value of t, give the co-ordinates of the cusp.

It is, in general, not possible to find any point through which four consecutive lines of the system pass, since the four equations

$$T = 0, \quad \frac{dT}{dt} = 0, \quad \frac{d^2T}{dt^2} = 0, \quad \frac{d^3T}{dt^3} = 0,$$

are more than sufficient to determine the three variables t, x, y. If we eliminate these three quantities between the four equations we shall have an equation between constants only, which will be the condition for the existence of such points. Such points, if they exist, would be triple points on the curve U, all the tangents at which would coincide.

104. To illustrate the preceding theory, we take the example where T is a rational function of the parameter t; for instance,

$$at^n + nbt^{n-1} + \frac{n(n-1)}{1.2}ct^{n-2} + \&c. = 0,$$

a, b, c, &c. being linear functions of the co-ordinates. On substituting in the equation the co-ordinates of any point, and solving for t, we get the values corresponding to the tangents which can be drawn from that point; and since, in the present instance, the resulting equation is of the n^{th} degree in t, it is plain that the curve U is of the n^{th} class. For any point whose co-ordinates satisfy the equations $T = 0$, $\frac{dT}{dt} = 0$, the equation T has two equal roots, or two of the tangents which can be drawn from it to the curve U coincide. Such a point is then *on* the curve U; for every two consecutive tangents intersect on the curve. To determine the point of contact of any tangent, it is convenient to join to the equation $\frac{dT}{dt} = 0$, the equation $nT - t\frac{dT}{dt} = 0$, which is only of the $n - 1^{st}$ degree. The degree of U is (see Note on Elimination) $2(n-1)$. The equation T is then not capable of expressing the most general curves of the n^{th} class, their degree being $n(n-1)$; or, in other words, the curve U must possess multiple points.

A cusp being a point at which three consecutive tangents in-

tersect, for such a point the equation T has three roots equal. The conditions that this shall be the case are

$$\mathrm{T} = 0, \quad \frac{d\mathrm{T}}{dt} = 0, \quad \frac{d^2\mathrm{T}}{dt^2} = 0,$$

which may be reduced to

$$at^{n-2} + (n-2)\,bt^{n-3} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,ct^{n-4} + \&c. = 0.$$

$$bt^{n-2} + (n-2)\,ct^{n-3} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,dt^{n-4} + \&c. = 0.$$

$$ct^{n-2} + (n-2)\,dt^{n-3} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,et^{n-4} + \&c. = 0.$$

From these three equations if we eliminate x and y, which enter in the first degree into each, we shall have an equation of the degree $3(n-2)$ in t. The curve will then have $3(n-2)$ cusps.

A double point on U is a point at which two distinct pairs of consecutive tangents intersect. If, then, we form the equation (see Note on Elimination) that the equation T should have two distinct pairs of equal roots, this, together with the equations

$$\mathrm{T} = 0, \quad \frac{d\mathrm{T}}{dt} = 0,$$

make up three equations, from which, as before, we may eliminate x and y, and obtain an equation in t, whose roots correspond to the tangents at the double points on the curve.

Knowing the degree and class of the curve U, and the number of its cusps, we have the number of its double points,

$$2\delta = 2(n-1)\,(2n-3) - n - 9(n-2).$$

Hence $$\delta = 2(n-2)\,(n-3).*$$

105. The points of inflexion of the curve U can be determined from the consideration that at such a point the tangent coincides with the consecutive tangent. Hence $\mathrm{T} = 0$ and $\frac{d\mathrm{T}}{dt} = 0$ must represent the same right line. The curve, then, which we

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* I had derived this value of δ directly as a particular case of a more general theorem (see Cambridge and Dublin Mathematical Journal, vol. iii. p. 170), but I do not at present remember the proof by which I obtained that theorem.

are considering has in general no points of inflexion, since two conditions must be fulfilled, in order that two equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

should represent the same right line $\left(\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}\right)$, and if t be determined so as to satisfy one of these, it will, in general, not satisfy the other.

Double tangents occur when the equation $T = 0$ represents the same right line for different values of t. Take, for example, the case of the third degree: it is required to determine $t_1\, t_2$, so that the same right line shall be represented by

$$at_1^3 + 3bt_1^2 + 3ct_1 + d = 0, \text{ and } at_2^3 + 3bt_2^2 + 3ct_2 + d = 0.$$

Now d can be expressed in terms of a, b, c; $d = aA + bB + cC$. We have then the equations

$$\frac{t_1^3 + A}{t_2^3 + A} = \frac{3t_1^2 + B}{3t_2^2 + B} = \frac{3t_1 + C}{3t_2 + C}.$$

Equate the second and third of these, divide by $t_1 - t_2$, denote $t_1 + t_2$ by p, $t_1 t_2$ by q; and we have

$$3q + Cp - B = 0.$$

Similarly, from the first and third,

$$3pq + C(p^2 - q) - 3A = 0;$$

whence $$Bp - Cq - 3A = 0.$$

Since then p and q are determined by equations of the first degree, the curve has but one double tangent.

The number of double tangents, in general, as determined by Plücker's theory, from the consideration that this curve of the n^{th} class is reduced by double tangents alone to the $2(n-1)^{st}$ degree, is $\frac{(n-1)\,(n-2)}{2}$.

106. The following examples of envelopes, which often occur in practice, may serve to illustrate the preceding theory.

(1.) To find the envelope of $A\mu^n + B\mu^p + C = 0$, we have the two equations

$$nA\mu^{n-p} + pB = 0,$$

$$(n-p)\mu^p B + nC = 0;$$

whence eliminating μ, we have

$$n^n A^p C^{n-p} \pm p^p (n-p)^{n-p} B^n = 0,$$

where the sign + is to be used when n is odd, and − when it is even.

(2.) To find the envelope of $A\cos^m\theta + B\sin^m\theta = C$, where θ is variable, A, B, C any functions of the co-ordinates.

$$\frac{dT}{d\theta} = -A\cos^{m-1}\theta\sin\theta + B\sin^{m-1}\theta\cos\theta = 0,$$

or

$$\tan\theta = \frac{A^{\frac{1}{m-2}}}{B^{\frac{1}{m-2}}};\ \cos\theta = \frac{B^{\frac{1}{m-2}}}{\sqrt{\left(A^{\frac{2}{m-2}} + B^{\frac{2}{m-2}}\right)}};\ \sin\theta = \frac{A^{\frac{1}{m-2}}}{\sqrt{\left(A^{\frac{2}{m-2}} + B^{\frac{2}{m-2}}\right)}}.$$

Substituting these values, we get, after reductions, for the envelope,

$$A^{\frac{2}{2-m}} + B^{\frac{2}{2-m}} = C^{\frac{2}{2-m}}.$$

Conversely, any tangent to the curve $x^n + y^n = a^n$ may be expressed by

$$x\cos^{\frac{2(n-1)}{n}}\theta + y\sin^{\frac{2(n-1)}{n}}\theta = a.$$

(3.) As an instance of the application of (2), we give, *To find the envelope of a line of constant length moving between two rectangular lines.*

Let the given length be c; then the intercepts on the sides being $c\cos\theta$, $c\sin\theta$, the equation of the line is

$$\frac{x}{c\cos\theta} + \frac{y}{c\sin\theta} = 1,$$

of the form discussed in the last paragraph, and whose equation is therefore

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

(4.) (2) is itself a particular case of the problem given, *Conics*, p. 241, *To find the envelope of*

$$(Ax)^m + (By)^m + C^m = 0;$$

the variables x, y being also connected by the relation

$$(ax)^n + (by)^n + c^n = 0.$$

We have

$$A^m x^{m-1}dx + B^m y^{m-1}dy = 0,\ a^n x^{n-1}dx + b^n y^{n-1}dy = 0.$$

Hence

$$A^m b^n x^{m-n} = B^m a^n y^{m-n};$$

and, therefore,

$$(\mathrm{A}b)^{\frac{mn}{m-n}}(\mathrm{A}x)^m = (\mathrm{B}a)^{\frac{mn}{m-n}}(\mathrm{B}y)^m \text{ and } (\mathrm{A}b)^{\frac{mn}{m-n}}(ax)^n = (\mathrm{B}a)^{\frac{mn}{m-n}}(by)^n.$$

Hence

$$\mathrm{A}^m x^m = \frac{-\mathrm{C}^m\left(\dfrac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}}}{\left(\dfrac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}} + \left(\dfrac{b}{\mathrm{B}}\right)^{\frac{mn}{m-n}}} \text{ and } a^n x^n = \frac{-c^n\left(\dfrac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}}}{\left(\dfrac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}} + \left(\dfrac{b}{\mathrm{B}}\right)^{\frac{mn}{m-n}}}.$$

Raising the first of these equations to the n^{th}, the second to the m^{th} power, and dividing one by the other, we have

$$\left\{\left(\frac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}} + \left(\frac{b}{\mathrm{B}}\right)^{\frac{mn}{m-n}}\right\}^{m-n} = (-1)^{m-n}\left(\frac{c}{\mathrm{C}}\right)^{mn},$$

or

$$\left(\frac{a}{\mathrm{A}}\right)^{\frac{mn}{m-n}} + \left(\frac{b}{\mathrm{B}}\right)^{\frac{mn}{m-n}} + \left(\frac{c}{\mathrm{C}}\right)^{\frac{mn}{m-n}} = 0.$$

107. We add, as a further example, the method of finding the equation of the curve reciprocal to a given one; or, as we may otherwise express it, the method of transforming an equation from point to line co-ordinates, and *vice versâ*. For symmetry we take the reciprocal with regard to the curve $x^2 + y^2 + z^2 = 0$. This will be made to coincide with ordinary circular polars, by taking $z^2 = -k^2$. The polar, then, of any point on the curve is

$$xx' + yy' + zz' = 0,$$

and the problem is reduced to finding the envelope of this line when $x'y'z'$ satisfy the equation of the curve. Now, precisely as at *Conics*, p. 267, we eliminate z' by multiplying the whole equation by z^n, the equation then becomes a homogeneous function of $x'y'$, and the condition that this equation in $x' : y'$ should have equal roots is the equation of the reciprocal curve. The only theoretical inconvenience of this method is, that the result is the equation of the reciprocal curve multiplied by the irrelevant factor $z^{n(n-1)}$. This, however, will not be found in practice to cause any difficulty.

108. To illustrate this method, we give the process of finding the reciprocal of a curve of the third degree given by its general equation, which we write:

$$a_1x^3 + b_2y^3 + c_3z^3 + 6dxyz + 3a_2x^2y + 3a_3x^2z + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y = 0.*$$

The reciprocal is now the condition that the following equation should have equal roots:

$$(a_1z^3 - 3a_3z^2x + 3c_1zx^2 - c_3x^3)\,x_1^3 + (b_2z^3 - 3b_3z^2y + 3c_2zy^2 - c_3y^3)\,y_1^3 + 3(a_2z^3 - 2dz^2x - a_3z^2y + 2c_1zxy + c_2zx^2 - c_3x^2y)\,x_1^2y_1 + 3(b_1z^3 - 2dz^2y - b_3z^2x + 2c_2zxy + c_1zy^2 - c_3xy^2)\,x_1y_1^2 = 0.$$

The condition that this should have equal roots is (see Note on Elimination)

$$(a_1z^3 - 3a_3z^2x + 3c_1zx^2 - c_3x^3)^2\,(b_2z^3 - 3b_3z^2y + 3c_2zy^2 - c_3y^3)^2 + 4(a_1z^3 - 3a_3z^2x + 3c_1zx^2 - c_3x^3)\,(b_1z^3 - 2dz^2y - b_3z^2x + 2c_2xyz + c_1y^2z - c_3xy^2)^3 + 4(b_2z^3 - 3b_3z^2y + 3c_2zy^2 - c_3y^3)\,(a_2z^3 - 2dz^2x - a_3z^2y + 2c_1xyz + c_2x^2z - c_3x^2y)^3 - 3(b_1z^3 - 2dz^2y - b_3z^2x + 2c_2xyz + c_1y^2z - c_3xy^2)^2\,(a_2z^3 - 2dz^2x - a_3z^2y + 2c_1xyz + c_2x^2z - c_3x^2y)^2 - 6(a_1z^3 - 3a_3z^2x + 3c_1zx^2 - c_3x^3)\,(b_2z^3 - 3b_3z^2y + 3c_2zy^2 - c_3y^3)\,(a_2z^3 - 2dz^2x - a_3z^2y + 2c_1zxy + c_2x^2z - c_3x^2y)\,(b_1z^3 - 2dz^2y - b_3z^2x + 2c_2xyz + c_1y^2z - c_3xy^2) = 0.$$

From these it is not very difficult to pick out the coefficient of any power of the variables; remembering that the result is to be divided by z^6 in order to give the equation of the reciprocal curve. Thus the coefficient of z^6 will be

$$a_1^2b_2^2 + 4a_1b_1^3 + 4b_2a_2^3 - 3b_1^2a_2^2 - 6a_1a_2b_1b_2.$$

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* The above is the form in which I shall generally write the equation of the third degree. The coefficients are introduced on account of the symmetry which they give to the differentials with regard to xyz. With regard to the letters, when we form any function into which xyz enter symmetrically (such as HU, or the equation of the reciprocal curve, &c.), the coefficient of any power of x may be deduced from that of the corresponding power of y or z, by a suitable interchange of letters. It is desirable, therefore, to use such a notation as will plainly indicate of what power each letter is the multiplier, and thus readily show how the required interchange is to be made. Some modern writers use for the coefficient of $x^\alpha y^\beta z^\gamma$ the notation $a_{\alpha\beta\gamma}$, or simply $(\alpha, \beta, \gamma,)$. Thus the equation could be written $(3,0,0)\ x^3 + 3\ (2,1,0)\ x^2y + $ &c. Others write the variables x_1, x_2, x_3, and then write the equation $(1,1,1)\ x_1x_1x_1 + 3(1,1,2)\ x_1x_1x_2 + $ &c. This notation, however, is so cumbrous, that in practice it must, I think, be found very inconvenient. In the notation which I adopt, the letters a, b, c, and the numbers $1, 2, 3$, belong to x, y, z respectively, and any coefficient is the multiplier of the square of the variable indicated by the letter, into the variable indicated by the suffix; thus a_2 multiplies x^2y, &c.

The coefficient then of x^6 can be written down from symmetry, as explained in the previous note,

$$b_2^2c_3^2 + 4c_3b_3^3 + 4b_2c_2^3 - 3b_3^2c_2^2 - 6b_2b_3c_2c_3;$$

that of y^6 will be

$$a_1^2c_3^2 + 4a_3^3c_3 + 4a_1c_1^3 - 3a_3^2c_1^2 - 6a_1a_3c_1c_3.$$

We add all the coefficients whose form is really distinct. The other coefficients may be obtained from those we give, by interchange of letters.

$$6z^5x(a_2^2b_1b_3 + a_1a_2b_2b_3 + 2a_1b_1b_2d + 2a_2b_1^2d + 3a_2a_3b_1b_2 - a_1a_3b_2^2 - 2b_1^3a_3 - 2a_1b_1^2b_3 - 4a_2^2b_2d).$$

$$3z^4x^2(3a_3^2b_2^2 + 4b_1^3c_1 + 2a_1c_1b_2^2 + 12a_3b_3b_1^2 + 4a_1b_1b_3^2 + 4b_2c_2a_2^2 + 16a_2b_2d^2 - a_2^2b_3^2 - 8a_2b_1b_3d - 4b_1^2d^2 - 2a_2c_2b_1^2 - 2a_1b_1b_2c_2 - 6a_2a_3b_2b_3 - 6a_2b_1b_2c_1 - 12a_3b_1b_2d - 4a_1b_2b_3d).$$

$$6z^4xy(5a_1a_3b_2b_3 + 4a_1b_1^2c_2 + 4a_2^2b_2c_1 + 2a_1b_1b_3d + 2a_2a_3b_2d + 10a_3b_1^2d + 10a_2^2b_3d - 11a_2a_3b_1b_3 - 2a_1b_1b_2c_1 - 2a_1a_2b_2c_2 - 2a_2^2b_1c_2 - 2a_2b_1^2c_1 - 8a_2b_1d^2 - 4a_1b_2d^2 - 3a_3^2b_1b_2 - 3a_1a_2b_3^2).$$

$$2z^3x^3(6a_2b_1b_3c_2 + 3a_2b_1b_2c_3 + 9a_2b_2b_3c_1 + 3a_1b_2b_3c_2 + 9a_3b_1b_2c_2 + 18a_3b_2b_3d + 18b_1b_2c_1d + 6a_2b_3^2d + 6b_1^2c_2d + 12b_1b_3d^2 - a_1c_3b_2^2 - 2b_1^3c_3 - 2a_1b_3^3 - 9a_3c_1b_2^2 - 18a_3b_1b_3^2 - 18b_1^2c_1b_3 - 24a_2b_2c_2d - 16b_2d^3).$$

$$6z^3x^2y(2a_1b_3^2d + 2a_2a_3b_2c_2 + 4a_3b_2d^2 + 10a_2a_3b_3^2 + a_2c_3b_1^2 + 13a_2b_1b_3c_1 + 12a_2c_2b_1d + 8b_1d^3 + 9a_3b_1b_2c_1 + a_1b_1b_2c_3 + 6a_1b_2c_2d - 6a_3^2b_2b_3 - 4a_1b_2b_3c_1 - 5a_1b_1b_3c_2 - 8b_1^2c_1d - 11a_3b_1^2c_2 - 2a_3b_1b_3d - 4a_2^2b_3c_2 - 16a_2b_3d^2 - 2a_2^2b_2c_3 - 10a_2b_2c_1d).$$

$$6z^2x^2y^2\{4d^2(a_2c_2 + b_1c_1 + a_3b_3) - 8d^4 - 8d(a_1b_3c_2 + b_2c_1a_3 + c_3b_1a_2) + (a_1b_2c_1c_2 + a_1c_3b_1b_3 + b_2c_3a_2a_3) - 4a_1b_2c_3d + 18d(a_2b_3c_1 + a_3b_1c_2) + 4(a_2^2c_2^2 + b_1^2c_1^2 + a_3^2b_3^2) - 19(a_3b_3b_1c_1 + b_1c_1a_2c_2 + a_2c_2a_3b_3) + 5(a_2^2b_3c_3 + a_3^2b_2c_2 + b_1^2a_3c_3 + b_3^2a_1c_1 + c_1^2a_2b_2 + c_2^2a_1b_1)\}.^*$$

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* The problem of finding the equation of the reciprocal of a given curve may be either solved as an envelope or as a locus, for we may either find the envelope of the lines reciprocal to all the points of the given curve, or the locus of the points reciprocal to all its tangents. The former method appears to me much the easiest in practice; the latter, however, is that which has found most favour with geometers. Mr. Cayley, who, I believe, first published the solution of this problem in the case of the third degree, has, by this method, reduced it to elimination between seven linear equations. (Cambridge and Dublin Mathematical Journal, vol. i. p. 97; published in 1846.) Mr. Hesse, I believe independently, afterwards obtained the same solution of the problem (Crelle's Journal, vol. xxxvi. p. 172; published in 1848). Mr. Hesse, however, did not attempt to put his

109. If we write the equation of the fourth degree

$$a_1x^4 + b_2y^4 + c_3z^4 + 4a_2x^3y + 4a_3x^3z + 4b_1y^3x + 4b_3y^3z + 4c_1z^3x + 4c_2z^3y$$
$$+ 6dy^2z^2 + 6ez^2x^2 + 6fx^2y^2 + 12lx^2yz + 12mxy^2z + 12nxyz^2 = 0,$$

then the equation of its reciprocal will, in like manner, be found to be $S^3 = 27T^2$, where

$$\begin{aligned}S = {} & (b_2c_3 - 4b_3c_2 + 3d^2)x^4 + (c_3a_1 - 4a_3c_1 + 3e^2)y^4 + (a_1b_2 - 3a_2b_1 + 3f^2)z^4 \\ & + 4(b_3c_1 - b_1c_3 - 3nd + 3mc_2)x^3y + 4(b_1c_2 - b_2c_1 - 3md + 3nb_3)x^3z \\ & + 4(c_2a_3 - a_2c_3 - 3en + 3lc_1)y^3x + 4(a_2c_1 - a_1c_2 - 3le + 3na_3)y^3z \\ & + 4(b_3a_2 - b_2a_3 - 3mf + 3lb_1)z^3x + 4(a_3b_1 - a_1b_3 - 3lf + 3ma_2)z^3y \\ & + 6(a_1d - 2a_3m - 2a_2n + 2l^2 + ef)y^2z^2 + 6(b_2e - 2b_1n - 2b_3l \\ & + 2m^2 + fd)z^2x^2 + 6(c_3f - 2c_2l - 2c_1m + 2n^2 + de)x^2y^2 + 12(b_1c_1 \\ & + 2dl - eb_3 - fc_2 - mn)x^2yz + 12(c_2a_2 + 2em - fc_1 - da_3 - nl)y^2zx \\ & + 12(a_3b_3 + 2fn - da_2 - eb_1 - lm)z^2xy.\end{aligned}$$

To save room I only give the terms of T whose form is distinct. The others may be added by symmetrical interchange of letters.

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method into practice as Mr. Cayley has done, who has by this means actually worked out the equation of the reciprocal curve. The reader who may refer to his memoir will supply a figure of 2 omitted in the coefficients of $\xi\eta$, $\eta\zeta$, $\zeta\xi$ in the values of 2F, 2G, 2H, given p. 98, and correct the final result accordingly. Mr. Hesse has since published a simplified solution of the same question, still treating it as a locus (Crelle's Journal, vol. xli. p. 285), which for the third degree requires elimination only between four linear equations. But, as even this simplified method requires elimination between twelve linear equations in the case of the fourth degree, and is therefore nearly inapplicable to that case in practice, I have given the equation of the reciprocal of the fourth degree, in order more fully to show the superior facilities afforded by the method which I consider preferable.

I may be excused for adding, that I had worked out the equation of the reciprocal of the third degree, ten or twelve years ago. Not being then acquainted with M. Plücker's researches, I undertook the calculation with the view of observing in what cases the equation would lose dimensions, and it was thus, in fact, that I obtained my first knowledge of the effect of double points and cusps on the degree of reciprocal curves.

Mr. Cayley has remarked to me, that the equation of the reciprocal of

$$x^3 + y^3 + z^3 + 6dxyz = 0$$

(a form to which every equation of the third degree is reducible), can be easily made to exhibit the nine cusps. The tangent at any will be of the form

$$y - \theta z = 0,$$

(where θ is one of the cube roots of unity) and the co-ordinates of that cusp will be

$$x : y : z = d\theta^2 : \theta : 1.$$

$$T = (fa_1b_2 + 2fa_2b_1 - a_1b_1^2 - b_2a_2^2 - f^3)z^6 + 2(3f^2m + 3la_2b_2 + 2a_3b_1^2 + a_1b_1b_3 - ma_1b_2 - 2ma_2b_1 - fa_2b_3 - 2fa_3b_2 - 3flb_1)z^5x + (6feb_2 + 8ma_3b_2 + da_1b_2 + 2da_2b_1 + 4ma_2b_3 + 6fnb_1 + 6flb_3 + 12lmb_1 - 6na_2b_2 - a_1b_3^2 - 8a_3b_1b_3 - 9b_2l^2 - 6eb_1^2 - 3f^2d - 12fm^2)z^4x^2 + 2(9fa_3b_3 + lb_2a_3 - 10lb_3a_2 + ma_1b_3 - 10ma_3b_1 + 2na_1b_2 + 4na_2b_1 + 3feb_1 + 3fda_2 - 3flm - 3ea_2b_2 - 3da_1b_1 + 6b_1l^2 + 6a_2m^2 - 6nf^2)z^4xy + 2(a_2b_2c_2 + 2c_1b_1^2 + 2a_3b_3^2 + 6eb_1b_3 + 9b_2ln + 4m^3 + 6dfm - da_2b_3 - 2db_2a_3 - fb_1c_2 - 2fb_2c_1 - 6emb_2 - 3db_1l - 3fnb_3 - 6mnb_1 - 6mlb_3)z^3x^3 + 2(3elb_2 + da_1b_3 + 11db_1a_3 + 8na_2b_3 - 5nb_2a_3 + 15fmn + 12l^2b_3 + 3f^2c_2 + 12emb_1 + 3a_2b_2c_1 - 2a_2b_1c_2 - 12lnb_1 - 9dma_2 - 6lm^2 - 6lnb_1 - 3dfl - 3fb_1c_1 - 6ma_3b_3 - 15efb_3 - a_1b_2c_2)z^3x^2y + (a_1b_2c_3 - 20a_2b_3c_1 - 20a_3b_1c_2 + 2a_1b_3c_2 + 2b_2a_3c_1 + 2c_3a_2b_1 + 24lb_1c_1 + 24ma_2c_2 + 24na_3b_3 - 6elb_3 - 6enb_1 - 6dma_3 - 6dna_2 - 6flc_2 - 6fmc_1 - 3a_1d^2 - 3b_2e^2 - 3c_3f^2 - 30em^2 - 30dl^2 - 30fn^2 + 48def + 48lmn)z^2x^2y^2 + \&c.$$

From the form of this equation it appears that the reciprocal curve has twenty-four cusps, which lie on a curve of the fourth order, and consequently that the twenty-four tangents at points of inflexion on the given curve all touch a curve of the fourth class.

110. The equations of the reciprocals of curves of higher orders have only been obtained in a few particular cases. Thus, by Art. 106 (4), or by considering the intercepts made on the axes, by the tangent to the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

it is easy to see that the equation of the reciprocal is of the form

$$Ax^{\frac{m}{m-1}} + By^{\frac{m}{m-1}} = 1.$$

Another class of curves,

$$\rho^m = a^m \cos m\omega,$$

may serve to illustrate the use of polar co-ordinates, of which we have hitherto given but few examples.

If on any radius vector OP there be taken a portion OP′ equal to the consecutive radius vector OQ, then obviously $PP' = d\rho$; $P'Q = \rho d\omega$; and $\tan OPQ = \dfrac{\rho d\omega}{d\rho}$.

In the present example, taking the logarithmic differential,

$$\frac{d\rho}{\rho} = -\tan m\omega d\omega\,; \quad \frac{\rho d\omega}{d\rho} = -\cot m\omega\,;$$

and if θ be the acute angle made by the radius vector with the tangent, $\theta = 90° - m\omega$, and the perpendicular on the tangent

$$= \rho \sin\theta = \rho\cos m\omega.$$

The angle between this perpendicular and the radius vector $= m\omega$, and between the perpendicular and the line from which ω is measured $= (m+1)\,\omega$. But the radius vector of the reciprocal curve is the reciprocal of the perpendicular on the tangent; hence it is easy to see that the equation of the reciprocal curve is also of the form

$$\rho^m = a^m \cos m\omega,$$

the new m being equal to $-\dfrac{m}{m+1}$.

This family of curves includes several important species; for instance, the circle ($m = 1$), the right line ($m = -1$), the common lemniscata ($m = 2$), the equilateral hyperbola ($m = -2$), the cardioide ($m = \frac{1}{2}$), the focal parabola ($m = -\frac{1}{2}$), &c.

111. The problem to find the condition that the line $ax + by + cz$ should touch a given curve, is solved by exactly the same process by which the equation of the reciprocal curve is found; we eliminate z, and apply the condition that the resulting equation in x and y should have equal roots. Comparing this with the process of Art. 107, we see that the condition that this line should touch the curve is found by substituting in the equation of the reciprocal curve a, b, c, for x, y, z. (See *Conics*, p. 328.) This condition is then of the $n(n-1)^{st}$ degree in abc. The same thing appears from considering that this condition may be considered as a tangential equation of the curve (Art. 2). It is readily seen that this condition (and also the equation of the reciprocal curve) involve the coefficients of the original equation in the degree $2(n-1)$.

If we had this condition we should at once have the equation of the system of tangents drawn from a fixed point to the curve; for the equation of the line joining the points $\alpha\beta\gamma$, $\xi\eta\zeta$ is

$$x(\beta\zeta - \gamma\eta) + y(\gamma\xi - \alpha\zeta) + z(\alpha\eta - \beta\xi) = 0.$$

If then the point $\alpha\beta\gamma$ be fixed, and ξ, η, ζ the running co-ordinates of any point on a tangent through $\alpha\beta\gamma$, and if we form the condition that the line just written should touch the curve, we shall have the equation of all the tangents through $\alpha\beta\gamma$.

We see, then, that when we have the equation of the reciprocal curve we can obtain the equation of the system of tangents which can be drawn through $\alpha\beta\gamma$, by substituting in the equation of the reciprocal, for xyz, the quantities $\beta z - \gamma y$, $\gamma x - \alpha z$, $\alpha y - \beta x$. The equation of the system of tangents through $\alpha\beta\gamma$ must then be a homogeneous function of these quantities.

The form thus found for the equation of the system of tangents is sometimes more convenient than that found by the method of Art. 78. Thus the equation of the pair of tangents drawn from $\alpha\beta\gamma$ to the conic

$$Ax^2 + By^2 + Cz^2 = 0,$$

is obtained by this method in the form

$$\frac{(\beta z - \gamma y)^2}{A} + \frac{(\gamma z - \alpha x)^2}{B} + \frac{(\alpha y - \beta x)^2}{C} = 0,$$

which is readily seen to be equivalent to that obtained by the method of Art. 78, viz.,

$$(A\alpha x + B\beta y + C\gamma z)^2 = (Ax^2 + By^2 + Cz^2)\,(A\alpha^2 + B\beta^2 + C\gamma^2).$$

EVOLUTES.

112. One of the most important, and the earliest investigated class of envelopes, is that of the *evolutes* of curves. We have defined the evolute of a curve (*Conics*, p. 329) as the locus of the centres of curvature of the curve; but the evolute may also be defined as *the envelope of all the normals of the curve.* For the circle of curvature is that which passes through three consecutive points of the curve, and its centre is the intersection of perpendiculars at the middle points of the sides of the triangle formed by the points. But the lines joining the first and second, and the second and third points, are two consecutive tangents to the curve; and the perpendiculars to them just mentioned are two consecutive normals: the centre of curvature is therefore the intersection of two consecutive normals: and the locus of all the centres of curvature must be the same as the envelope of all the

normals. We add the investigation of the evolutes of the conic sections from this point of view.

(1.) To find the evolute of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The normal is (*Conics*, p. 161),

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2;$$

or, writing $x' = a\cos\phi$, $y' = b\sin\phi$,

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = c^2,$$

an equation of the class (Art. 106 (2)), and whose envelope is therefore

$$a^{\frac{2}{3}}x^{\frac{2}{3}} + b^{\frac{2}{3}}y^{\frac{2}{3}} = c^{\frac{4}{3}}.$$

(2.) The normal to a parabola is (*Conics*, p. 182)

$$p(y - y') + 2y'(x - x') = 0,$$

or

$$2y'^3 + (p^2 - 2px)y' - p^2y = 0,$$

an equation of the class (Art. 106 (1)), and whose envelope, considering y' variable, is

$$2(p - 2x)^3 + 27py^2 = 0.$$

(3.) *To find the evolute of the semicubical parabola*, $py^2 = x^3$.

The equation of the normal is

$$3x'^2(y - y') + 2py'(x - x') = 0.$$

Substitute for y' in terms of x' from the equation of the curve, divide by $x'^{\frac{3}{2}}$, and (putting $x'^{\frac{1}{2}} = \mu$) the equation becomes

$$3\mu^4 + 2p\mu^2 - 3p^{\frac{1}{2}}y\mu - 2px = 0,$$

whose envelope is

$$p(p - 18x)^3 = (54px + \tfrac{729}{16}y^2 + p^2)^2.$$

(4.) *To find the evolute of the cubical parabola*, $p^2y = x^3$.

The equation of the normal is

$$3x'^2(y - y') + p^2(x - x') = 0,$$

or

$$3x'^5 - 3p^2yx'^2 + p^4x' - p^4x = 0.$$

Now the envelope of

$$at^5 + 10dt^2 + 5et + f = 0$$

is

$$(af^2 - 12d^2e)^2 + 128(2e^2 - 3df)(ae^3 - adef - 9d^4) = 0.$$

Therefore, the envelope in the present case is

$$3p^2\left(x^2 - \tfrac{9}{125}y^2\right)^2 + \tfrac{128}{125}\left(\tfrac{2}{5}p^2 - \tfrac{9}{2}xy\right)\left(\frac{p^4}{5} - \tfrac{3}{2}p^2xy - \tfrac{243}{400}y^4\right) = 0.$$

(5.) As a further example, we give, to find the evolute of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

This curve belongs to the class $x^m + y^m = a^m$, and therefore (Art. 106 (2)) at any point of it we may write $x' = a\cos^3\phi$, $y' = a\sin^3\phi$, and the tangent at the point will be

$$\frac{x}{\cos\phi} + \frac{y}{\sin\phi} = a.$$

The perpendicular to this line at the point $x'y'$ will be

$$x\cos\phi - y\sin\phi = a\cos 2\phi,$$

or

$$(x+y)(\cos\phi - \sin\phi) + (x-y)(\cos\phi + \sin\phi) = 2a(\cos^2\phi - \sin^2\phi),$$

$$\frac{x+y}{\sin(\phi+45^\circ)} + \frac{x-y}{\cos(\phi+45^\circ)} = 2^{\frac{3}{2}}a.$$

The envelope is therefore (Art. 106),

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.*$$

113. We can readily obtain a general expression for the co-ordinates of the centre of curvature and for the radius of curvature. We use Cartesian co-ordinates in this and the next Articles. If $\alpha\beta$ be the co-ordinates of any point on the tangent, x and y those of its point of contact, the equation of the tangent is

$$(\alpha - x)\,dy = (\beta - y)\,dx;$$

where $\dfrac{dy}{dx}$ is to be found from the equation of the curve; for the tangent passes through the point xy, and makes an angle with the axis of x, whose tangent $= \dfrac{dy}{dx}$ (Art. 39). The normal, then, being a perpendicular to this at the point xy has for its equation

$$(\alpha - x)\,dx + (\beta - y)\,dy = 0.$$

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* A geometrical proof, and an extension of this theorem, will be given in a future Chapter. I perceive, however, that this evolute has been already given (Gregory's Examples by Walton, p. 195).

The point of contact of this line with its envelope is found by combining with this equation that obtained by differentiation, viz.,

$$(\alpha - x)\, d^2x + (\beta - y)\, d^2y = dx^2 + dy^2.$$

Hence

$$\alpha - x = \frac{dy\,(dx^2 + dy^2)}{d^2x dy - d^2y dx}; \quad \beta - y = \frac{-\,dx\,(dx^2 + dy^2)}{d^2x dy - d^2y dx};$$

and the radius of curvature is given by the equation

$$R = \sqrt{\{(\alpha - x)^2 + (\beta - y)^2\}} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2x dy - d^2y dx}.$$

These values, which have been obtained for the intersection of two consecutive normals, might have been found for the same point, considered as the centre of curvature. The equation of any circle is

$$(\alpha - x)^2 + (\beta - y)^2 = R^2,$$

where $\alpha\beta$ are the co-ordinates of the centre, x and y those of the point common to the curve and the circle; but since the circle touches the curve, $\frac{dy}{dx}$ is the same for both, and, therefore, differentiating, we have

$$(\alpha - x)\, dx + (\beta - y)\, dy = 0,$$

where we may give to $\frac{dy}{dx}$ the value obtained from the equation of the curve. Again, since the circle osculates the curve, we may differentiate a second time, and shall have

$$(\alpha - x)\, d^2x + (\beta - y)\, d^2y = dx^2 + dy^2;$$

but these are the same two equations which we have obtained already from other considerations.

114. The same investigation may with advantage be conducted in a different form.

Let us, for abbreviation, put, as at Art. 90,

$$\frac{dU}{dx} = L,\ \frac{dU}{dy} = M; \quad \frac{d^2U}{dx^2} = A,\ \frac{d^2U}{dxdy} = B,\ \frac{d^2U}{dy^2} = C.$$

Then $L dx + M dy = 0$, and the equation of the normal is

$$M(\alpha - x) = L(\beta - y).$$

Differentiating, we have

$$(\alpha - x)(Bdx + Cdy) - Mdx = (\beta - y)(Adx + Bdy) - Ldy,$$

or $$(\alpha - x)(BM - CL) - (\beta - y)(AM - BL) - L^2 - M^2 = 0.$$

Hence

$$\alpha - x = -\frac{L.(L^2 + M^2)}{AM^2 - 2BLM + CL^2}, \quad \beta - y = -\frac{M.(L^2 + M^2)}{AM^2 - 2BLM + CL^2},$$

$$R = \pm \frac{(L^2 + M^2)^{\frac{3}{2}}}{AM^2 - 2BLM + CL^2}.$$

This expression can be made to assume a more symmetrical form by introducing the linear unit z, so as to give the equation the trilinear form. Then, as we have proved (Art. 90),

$$AM^2 - 2BLM + CL^2 = \frac{Hz^2}{(n-1)^2},$$

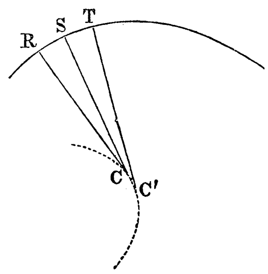
and $$R = \pm \frac{(n-1)^2 (L^2 + M^2)^{\frac{3}{2}}}{z^2 H}.$$

For any point whose co-ordinates satisfy the equation $H = 0$, the radius of curvature becomes infinite, and the centre of curvature at an infinite distance. This will take place when three consecutive points of the curve are on a right line, for then the circle through them becomes a right line, and its centre becomes at an infinite distance. We might then, from this value of the radius of curvature, arrive, independently of Art. 83, at the conclusion that the intersections of U and H are points of inflexion.

The double sign in the value of the radius of curvature is analogous to that in the value of the perpendicular on a right line (*Conics*, p. 20); and, of course, if we agree to use the sign + when the radius of curvature, and therefore the concavity of the curve, is turned in one direction, we must use the sign − when it is turned in the opposite direction. Since every algebraic function changes sign in passing through zero, we see that at a point of inflexion the radius of curvature changes sign, and that as we pass such a point the concavity of the curve changes to convexity, and *vice versâ* (see fig. p. 35). At a double point the radius of curvature assumes the form $\frac{0}{0}$, and its value must be determined by the ordinary rules in such cases. In fact, each branch of the

curve has its own curvature at the point. At a cusp it will be found that the radius of curvature vanishes.

115. *The length of any arc of the evolute is equal to the difference of the radii of curvature at its extremities.*



For, draw any three consecutive normals to the original curve: let C be the point of intersection of the first and second, C′ of the second and third; then since, ultimately, CR = CS, C′S = C′T; CC′, which is the increment of the arc of the evolute, is also the increment of the radius of curvature.

Hence, if a flexible thread be supposed rolled round the evolute, and wound off, any point of it will describe an *involute* of the curve CC′; that is, a curve of which CC′ is the evolute. It was from this point of view that Huyghens, the inventor of evolutes, first considered them, and it was hence that the name *evolute* was given. We shall in a future Chapter show how the problem is to be treated, "Given the equation of the evolute to find that of any of its involutes."

116. We add some other considerations respecting the evolutes of curves. Let it be required to find the *class* of the evolute of a curve of the m^{th} degree. This is evidently the same as the number of normals which can be drawn to the curve from any given point. Now, if in the equation of the normal $M(\alpha - x) = L(\beta - y)$, we suppose α, β given, x, y variable, we shall have the equation of a curve of the m^{th} degree, whose intersection with the given curve determines m^2 points, the normals at any one of which will pass through $\alpha\beta$. Or, the same thing may appear geometrically, thus: the number of normals is, by the law of continuity, the same, whatever be the point through which they pass; we may, therefore, content ourselves with examining the case where the point is at infinity. But the number of normals which can be drawn parallel to a given line is equal to the number of tangents which can be drawn parallel to a given line, that is, to the degree of the reciprocal of the curve $(m^2 - m)$. This is not the whole number of normals which can be drawn through a point at infi-

nity; for the reader will readily perceive that when a point in the curve is at infinity, the normal at it will lie altogether at infinity, and therefore the m normals corresponding to the m points of the curve at infinity will also pass through the given point; we must then add this number to that already obtained, and we find that *the number of normals which can be drawn to the curve from a given point is equal to the sum of the degrees of the curve and its reciprocal*, and therefore $= m^2$, if the curve have no double points.

If the line at infinity were a tangent to the curve, then the number of finite tangents which can be drawn through a point at infinity is plainly one less than in the general case, and therefore the number of normals is one less than in the general case. Thus four normals can be drawn from a given point to an ellipse or hyperbola, but only three to a parabola.

117. The line $y = mx$ being perpendicular to the line $y = -\frac{1}{m}x$, when $m = \sqrt{(-1)}$, the line and its perpendicular coincide. In fact, it will be seen that when the tangent of an angle is $\sqrt{(-1)}$, its sine and cosine are infinite, and when a line makes an infinite angle with the axis of x, its perpendicular can do no more. When, therefore, the curve passes through the two imaginary circular points at infinity, the normals at these points will coincide with the tangents at them, and not, in this case, with the line at infinity; the normals at these points will then, in general, not pass through any other point at infinity; and it appears from the proof of the last Article that the class of the evolute is diminished by two. If these two imaginary points be double points on the curve, it is not difficult to trace the effect on the class of the evolute.

118. Since, in projection, the line answering to the normal of a curve is not normal to its projection, it may be of use to mention what is the more general relation in which that of normal is included. Since the tangent and normal cut harmonically the line joining the two circular points at infinity (*Conics*, p. 312), we see that if we take a fourth harmonic to the tangent, and the lines joining its point of contact to two fixed points, we shall have a

line, which may be called the quasi-normal, and its envelope will be a quasi-evolute. Take, for example, the curve discussed (Art. 104), the tangent to which is

$$T = a\lambda^m + mb\lambda^{m-1}\mu + \&c. = 0,$$

then the quasi-normal will be the fourth harmonic to the tangent

$$\lambda\frac{dT}{d\lambda} + \mu\frac{dT}{d\mu} = 0,$$

and to the two lines

$$\left(\frac{dT}{d\mu}\right)_1\left(\frac{dT}{d\lambda}\right) - \left(\frac{dT}{d\lambda}\right)_1\left(\frac{dT}{d\mu}\right) = 0,$$

$$\left(\frac{dT}{d\mu}\right)_2\left(\frac{dT}{d\lambda}\right) - \left(\frac{dT}{d\lambda}\right)_2\left(\frac{dT}{d\mu}\right) = 0,$$

where $\left(\frac{dT}{d\lambda}\right)_1\left(\frac{dT}{d\lambda}\right)_2$ mean the result of substituting the co-ordinates of the two fixed points in $\frac{dT}{d\lambda}$. This fourth harmonic will be seen to be (*Conics*, Art. 55)

$$\left\{2\mu\left(\frac{dT}{d\mu}\right)_1\left(\frac{dT}{d\mu}\right)_2 + \lambda\left(\frac{dT}{d\mu}\right)_1\left(\frac{dT}{d\lambda}\right)_2 + \lambda\left(\frac{dT}{d\mu}\right)_2\left(\frac{dT}{d\lambda}\right)_1\right\}\frac{dT}{d\lambda}$$
$$= \left\{2\lambda\left(\frac{dT}{d\lambda}\right)_1\left(\frac{dT}{d\lambda}\right)_2 + \mu\left(\frac{dT}{d\mu}\right)_1\left(\frac{dT}{d\lambda}\right)_2 + \mu\left(\frac{dT}{d\mu}\right)_2\left(\frac{dT}{d\lambda}\right)_1\right\}\frac{dT}{d\mu}.$$

Since this equation involves $\lambda : \mu$ in the $3m - 2$ degree, this will be the class of the curve, as otherwise appears from the theory of Art. 116, since the curve enveloped by T is of the order $2m - 2$, and class m.

119. We proceed next to examine the *degree* of the evolute; and by the law of continuity it suffices to examine the number of points in which the line at infinity can meet the evolute. Now, if two consecutive normals to the original curve be parallel, the corresponding tangents will coincide; the points at infinity, therefore, on the evolute arise, in general, from the points of inflexion on the given curve. But to these must be added those arising from points of infinity on the given curve, for it has been shown (Art. 116) that these also give rise to points at infinity on the evolute. But we say, moreover, that these will be cusps on the evolute, at which the line at infinity is the tangent. This will

appear more readily by considering the projection of the figure. Let OO′ be the projection of the line at infinity, M one of the points in which the curve

meets OO′, LN the two adjacent points, M′ the fourth harmonic to OMO′; then the line answering to the normal at M will, as we have seen, be OO′; those answering to the normals at LN will be LM′, NM′; hence M′ is a point through which three consecutive tangents to the evolute pass, and is therefore a cusp at which OO′ is the tangent. Since then the tangent at a cusp meets the curve in three consecutive points, the m points at infinity of the given curve give rise to the same number of cusps on the evolute which are met by the line at infinity in $3m$ points. If we add these to those already obtained, we find the degree of the evolute $= \iota + 3m = 3m(m-1)$ when the curve has no multiple points.

If the curve pass through the two circular points at infinity, we have seen that these give rise to no points at infinity on the evolute, and therefore its degree will be less by six.

If the line at infinity touch the curve, the point N on the last figure will lie on the line at infinity; two tangents to the evolute will then coincide with this line, and we shall have a point of inflexion on the evolute at infinity. As this takes the place of two cusps, which we have when the line at infinity meets the curve in distinct points, the degree of the evolute is reduced by three.

120. There will in general be no points of inflexion on the evolute. For if there be such a point, two consecutive tangents to the evolute (normals to the curve) must coincide; but it is plain, on considering the figure, that two consecutive normals cannot coincide unless the corresponding tangents coincide with their normals and with each other. There cannot then be a finite point of inflexion on the evolute, save in the exceptional case, when the original curve has a point of inflexion whose tangent passes through one of the circular points at infinity (Art. 117).

Having then the degree, the class of the evolute, and the number of its points of inflexion, we have (by Art. 100) sufficient to determine its other singularities; we shall, however, give an

independent investigation of the number of cusps on the evolute, which will serve to verify our other conclusions.

A cusp on the evolute will in general take place when three consecutive tangents to the evolute (normals to the curve) intersect in a point. To find the condition that this should happen, we must join to the equations of Art. 114, viz.,

$$M(\alpha - x) = L(\beta - y),$$

$$(\alpha - x)(BM - CL) - M^2 = (\beta - y)(AM - BL) + L^2,$$

that obtained by differentiating again; or writing

$$\frac{d^3U}{dx^3} = a_1, \quad \frac{d^3U}{dx^2dy} = a_2, \quad \frac{d^3U}{dxdy^2} = b_1, \quad \frac{d^3U}{dy^3} = b_2,$$

$$\begin{aligned}(\alpha - x)\{a_2M^2 - 2b_1LM + b_2L^2 + (B^2 - AC)M\}& \\ - (\beta - y)\{a_1M^2 - 2a_2LM + b_1L^2 + (B^2 - AC)L\}& \\ = 2L(AM - BL) + 2M(BM - CL).&\end{aligned}$$

Substituting in this equation the values of $\alpha - x$, $\beta - y$, found from the first two, as in Art. 114, we obtain for the condition that three consecutive normals should intersect in a point (or that four consecutive points of the curve should lie on a circle),

$$\begin{aligned}(L^2 + M^2)\{b_2L^3 - 3b_1L^2M + 3a_2LM^2 - a_1M^3\}& \\ + 2(AM^2 - 2BLM + CL^2)\{(A - C)LM + B(M^2 - L^2)\} = 0.&\end{aligned}$$

This takes a simpler form by introducing the linear unit z, giving the equation the trilinear form, and employing the values found for $b_2L^3 - 3b_1L^2M + 3a_2LM^2 - a_1M^3$, $AM^2 - 2BLM + CL^2$, in Arts. 90, 94, when we have

$$(L^2 + M^2)\left(M\frac{dH}{dy} - L\frac{dH}{dx}\right) + 2H\{(A - C)LM + B(M^2 - L^2)\} = 0.$$

Remembering that H is of the degree $3(m - 2)$, A, B, C of the degree $m - 2$, and L, M of the degree $m - 1$, we see that this equation represents a curve of the degree $6m - 10$, whose intersection with the given curve determines $m(6m - 10)$ points, each of which gives rise to a cusp on the evolute. Add to these the m cusps, which we have seen exist at infinity; and the total number of cusps is found to be $m(6m - 9)$.

We have given (Art. 100) the general formula connecting the degree, class, cusps, and inflexions of a curve, viz.,

$$3(\mu - \nu) = \kappa - \iota;$$

but this is verified when

$$\mu = 3m(m-1), \quad \nu = m^2, \quad \kappa = m(6m-9), \quad \iota = 0.$$

Hence the theory of Sect. V. enables us to determine the number of double points and double tangents to the evolute, or to see in how many cases the same line can be doubly normal to the curve.

It appears from the values given that the degree and class are in general the same of the evolute of a curve and of its reciprocal.

It will readily appear that the locus of the extremity of the *polar subtangent* (see *Conics*, Art. 196) of any curve is the reciprocal of the evolute of the reciprocal curve. Thus this locus is a right line for the focal conics, since the evolute of the reciprocal then reduces to a point.

121. We conclude this part of the subject with an account of the manner in which radii of curvature and evolutes are to be found by polar co-ordinates. It is generally convenient to substitute an equation of the form $\rho = \phi(p)$ for the ordinary polar equation $\rho = \phi(\omega)$; where p is the perpendicular from the pole on the tangent, and is given by the equations

$$p = \rho \sin\theta; \ \tan\theta = \rho \frac{d\omega}{d\rho}. \quad \text{(Art. 110.)}$$

Let the distance from the pole to the centre of curvature be ρ_1, and the radius of curvature R, then (Euclid, II. 13)

$$\rho_1^2 = \rho^2 + R^2 - 2Rp.$$

If we pass to the consecutive point of the given curve, ρ_1 and R remain constant, and we have

$$\rho d\rho = R dp.$$

Example.—Let the curve be

$$\rho^m = a^m \cos m\omega.$$

Then (Art. 110) $$p = \rho \cos m\omega,$$

and $$\rho^{m+1} = a^m p;$$

whence, differentiating, we can easily see that $\rho^2 = (m+1)pR$.

In the general case, when we have expressed R in terms of ρ, p, if we eliminate ρ, p, between the equations

$$\rho = \phi p, \ \rho_1^2 = \rho^2 + R^2 - 2Rp,$$

and the equation, which is obviously true,

$$p_1^2 = \rho^2 - p^2,$$

we shall have the relation which subsists between the ρ_1 and p_1 of the evolute (Gregory's Examples, p. 197). We shall give examples of the application of this method when we come to treat of transcendental curves. It is not, however, always practicable to pass from the equation of the form $\rho = \phi(p)$ to the equation $\rho = \phi(\omega)$.

In the curve given in the last example, though the evolute cannot easily be expressed in general, its reciprocal can readily be found. For $p_1 = \rho \sin m\omega$, and the radius vector of the reciprocal curve is $\frac{1}{p_1}$, and (Art. 110) the angle made by p_1 with a line at right angles to the line whence ω is measured, is $(m+1)\omega$; hence the equation of the reciprocal of the evolute is of the form

$$\rho^m \cos \frac{m}{m+1}\,\omega \sin^m \frac{m}{m+1}\,\omega = a^m.$$

CAUSTICS.

122. As a further illustration of envelopes, we add some mention of caustics, the investigation of which, though suggested to mathematicians by the science of optics, belongs purely to the theory of curves. The subject has some historical interest, caustics being among the earliest questions, involving the problem of envelopes, actually discussed.*

If light be incident from any point on a curve, the reflected ray is found by drawing a line, making with the normal the same angle which is made with it by the incident ray: the envelope of all these reflected rays is the *caustic by reflexion.*

It is easy to form the general equation of the reflected ray. Let the equations of the tangent and normal at the point of incidence be $T = 0$, $N = 0$: then the equation of the incident ray is $T'N - TN' = 0$, where $T'N'$ are the results of substituting the coordinates of the radiant point in T and N: the reflected ray, then,

* The subject of caustics was introduced by Tschirnhausen, Acta Eruditorum, 1682, referred to by Gregory, Examples, p. 224.

which is the fourth harmonic to these three lines, will have for its equation

$$T'N + TN' = 0,$$

and the envelope can then be found by the preceding rules.

Example.—*To find the caustic by reflexion of a circle.*

The reflected ray is, by the preceding ($\alpha\beta$ being the co-ordinates of the radiant point, and the tangent and normal being

$$x\cos\theta + y\sin\theta - r, \text{ and } x\sin\theta - y\cos\theta),$$

$$(\alpha\cos\theta + \beta\sin\theta - r)(x\sin\theta - y\cos\theta) + (x\cos\theta + y\sin\theta - r)(\alpha\sin\theta - \beta\cos\theta) = 0,$$

or

$$(\alpha y + \beta x)\cos 2\theta + (\beta y - \alpha x)\sin 2\theta + r(x+\alpha)\sin\theta - r(y+\beta)\cos\theta = 0.$$

Now equations of this form,

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta = 0,$$

may, by putting $\theta = \phi + \alpha$, and determining α suitably, be reduced to the form

$$A_1\sin 2\phi + B_1\sin\phi + C_1\cos\phi = 0,$$

whose solution has been given already (Art. 106). This is the principle of Lagrange's solution of this problem (see Cambridge and Dublin Mathematical Journal, vol. ii. p. 236). Or the same equation, by putting $e^{\theta\sqrt{-1}} = z$ can be put into the form

$$(A - B\sqrt{-1})z^4 + (C - D\sqrt{-1})z^3 + (C + D\sqrt{-1})z + A + B\sqrt{-1} = 0,$$

whose envelope is (see Note on Elimination)

$$\{4(A^2 + B^2) - (C^2 + D^2)\}^3 = 27(AC^2 - AD^2 + 2BCD)^2.$$

Putting in the above values for A, B, C, D, we have

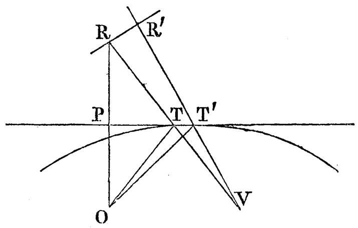
$$[4(\alpha^2 + \beta^2)(x^2 + y^2) - r^2\{(x+\alpha)^2 + (y+\beta)^2\}]^3 = 27(\beta x - \alpha y)^2(x^2 + y^2 - \alpha^2 - \beta^2)^2.*$$

123. Instead of finding directly the envelope of the reflected ray, M. Quetelet has given a method, which is more convenient in practice, of reducing the problem to that of evolutes; since the caustic would be sufficiently determined if we knew the curve of which it was the evolute.

"If with each point successively of the reflecting curve as

* See Gergonne's Annales, xvii. p. 128; Cambridge and Dublin Mathematical Journal, ii. p. 128.

centre, and its distance from the radiant point as radius, we describe a series of circles, the envelope of all these circles will be a curve, the evolute of which will be the caustic required." The following is a more convenient form of stating the same theorem: *If we let fall from the radiant point* O *the perpendicular* OP *on the tangent, and produce it, so that* PR = OP, *then the caustic is the evolute of the locus of* R.



For RT is evidently the direction of the reflected ray, and if we draw the consecutive ray, then, since OT, TV; OT′, T′V, make equal angles with TT′, OT + TV = OT′ + T′V (*Conics*, p. 290); therefore VR = VR′, and therefore VR is normal to the locus of R.

The locus of R is plainly a curve similar to the locus of P, and its equation can always be written down when the equation of the reciprocal of the given curve, with regard to O, is known; by substituting $\frac{2}{\rho}$ for ρ in the polar equation of that reciprocal. Thus the caustic by reflexion, of a circle, is the evolute of a curve whose equation (the radiant point being pole) is of the form

$$\rho = p(1 + e\cos\omega),$$

which, reduced to x and y co-ordinates, is of the fourth degree.

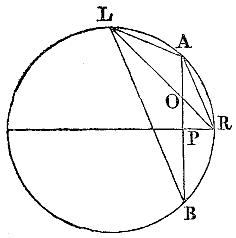
124. If light be incident from any point on a curve, the *refracted* ray is found by drawing a line, making with the normal an angle whose sine is in a constant ratio to that of the angle made with the normal by the incident ray, and the envelope of all these rays is the *caustic by refraction.*

M. Quetelet has reduced in like manner these caustics to evolutes by the following theorem, the truth of which it is easy to see. "If with each point successively of the refracting curve as centre, and a length *in a constant ratio to* its distance from the radiant point as radius, we describe a series of circles, the envelope of all these circles will be a curve whose evolute is the caustic by refraction." In fact, the method of infinitesimals readily shows that, in consequence of the law of refraction, the increments of

the incident and refracted rays are connected by the relation $md\rho + d\rho' = 0$, it follows, then, that if on the refracted ray produced, TR be taken $= m$OT, T′R′ $= m$OT′, then VR = VR′, and therefore the refracted ray is normal to the locus of R.

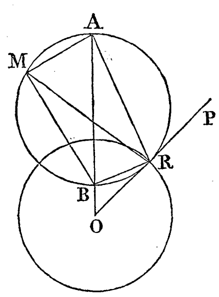
We add geometrical investigations of the two most interesting cases of caustics by refraction.

(1.) *To find the caustic by refraction of a plane refracting surface.*



Let fall a perpendicular on the plane, and produce it so that AP = PB; and let a circle be described through A, B, and the point of incidence R; let LR be the refracted ray; then obviously the angle ALB is bisected, and AL + LB : AB :: AL : AO :: sin AOL : sin ALO; but AOL is the angle which the refracted ray makes with the perpendicular to the surface, and ALO = BLO = BAR is the angle which the incident ray makes with the perpendicular: the ratio of AL + LB to AB is therefore given; the locus of L is an ellipse, of which A and B are the foci, to which LR is normal, and of which, therefore, the caustic is the evolute.

(2.) *To find the caustic by refraction of a circle.*



Let a circle be described through A, the radiant point, and R, the point of incidence, to touch OR; then the point B is given, since OA.OB = OR². The ratio RA : RB is by similar triangles equal to the given ratio OA : OR. The ratio RA : RM is equal to sin RBA : sin RBM; but RBA = PRA, the angle which the incident ray makes with the normal to the surface, and RBM = PRM, the angle which the refracted ray makes with the same normal; hence the ratio RA : RM is also given. Now since

$$AM.RB + MB.AR = RM.AB,$$

if we denote the distances of M from A and B by ρ, ρ', these distances are connected by the relation

$$\frac{RB}{RM}\rho + \frac{RA}{RM}\rho' = AB.$$

Now, a Cartesian oval is defined as the locus of a point whose distances from two given foci are connected by the relation $m\rho + n\rho' = c$; and it is proved precisely as at *Conics*, p. 290, that the normal to such a curve divides the angle between the focal radii into parts whose sines are in the ratio $m:n$. Hence the locus of M is a Cartesian oval, of which A and B are foci, and it is obvious that MR is normal to the locus, and therefore the caustic is the evolute of this curve.*

SECT. VII.—FOCI.

125. We have, in our section on Singular Points, showed the reader how to recognise all the points *on* curves which are specially distinguished from the rest: the example, however, of conic sections shows that there may be points not on the curve which play an important part in the theory of curves. All such points are included in the definition of a focus, given *Conics*, p. 237. We there defined a focus to be a point, such that the lines joining it to the two imaginary points on a circle at infinity shall both touch the curve. We extend this definition to curves in general; and we believe that it will be found that every point which has any special relation to any curve will be found either to be a singular point of the curve, or a focus of it.

The number of foci which a curve may have depends on the class of the curve. If n tangents can be drawn to the curve from one of the two imaginary points above-mentioned, the n^2 points of intersection of these with the tangents from the other imaginary point, will be the foci of the curve.

It is important to remark that n, and only n, of these foci, will be real; for if $A + B\sqrt{-1} = 0$ be the imaginary tangent from one of these points, $A - B\sqrt{-1} = 0$ will be a tangent from the other point, and will intersect the other in the real point AB: and plainly there can be but one real point on an imaginary line, since the line joining two real points is real.

There is no theoretical difficulty in analytically determining the foci when the equation of the curve is given. Determine the

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* This theorem is M. Quetelet's. The proof here given was communicated to me by Dr. Atkins.

condition that the line $x + y\sqrt{-1} = c$ should touch the curve; determine c so as to fulfil this condition, and if any of the values found for c be $a + b\sqrt{-1}$, then $x = a$, $y = b$, are the co-ordinates of one of the foci. Or, it appears from Art. 109, that if we substitute in the equation of the reciprocal curve, 1 for x, $\sqrt{-1}$ for y, and solve for z, the value of z will be of the form $-\{(x' + y'\sqrt{(-1)}\}$, where $x'y'$ are the co-ordinates of one of the foci.

126. The number of finite foci is diminished if the line at infinity touch the curve, for then only $n - 1$ finite tangents can be drawn from each of the two imaginary points, and therefore there are only $n - 1$ real finite foci. The n^{th} has gone off to an infinite distance, being, in fact, the point of contact of the line at infinity with the curve.

Thus we see why a parabola has but one focus.

So again, if the curve pass through the two imaginary points in question, since, of the tangents which can be drawn to a curve from a point on it, two coincide with the tangent at the point, we see that two of the foci of the curve will coincide. Thus, for a circle, the two foci coincide with the centre.

127. The manner in which M. Plücker has solved the problem of finding the foci does not substantially differ from the solution given in Art. 125; it may be well, however, to add it here in the form in which M. Plücker has given it.

Let $y - y' = p(x - x')$ be the equation of any line, then the condition that this should touch a curve of the n^{th} class (or the tangential equation of the curve) may be written

$$Ap^n + Bp^{n-1} + Cp^{n-2} + Dp^{n-3} + \&c. = 0,$$

where ABCD are functions of $x'y'$. Now if $p = \sqrt{-1}$, substituting this value, and equating to zero the real and imaginary parts of the equation, we have

$$A - C + E - \&c. = 0, \quad B - D + F - \&c. = 0;$$

the equations of two curves of the n^{th} degree, whose intersections will determine the n^2 foci. It is not difficult to see the geometric meaning of these curves. If $x'y'$ be given, by the theory of equations, $-\dfrac{B}{A}$ denotes the sum of the tangents of the angles which

the tangents to the curve through $x'y'$ make with the axis of x; $\frac{C}{A}$ denotes the sum of the products in pairs of those tangents, &c. And from the well-known formula for the tangent of the sum of several angles, the equation

$$B - D + F - \&c. = 0$$

expresses that the sum of the angles $= 0$, or $=$ some multiple of π, which the tangents to the curve through $x'y'$ make with the axis of x. And

$$A - C + E - \&c. = 0$$

expresses that the sum of these angles is equal to some odd multiple of $\frac{\pi}{2}$. Hence the locus of a point, such that the sum of the angles made with a fixed line by the tangents through it to a curve of the n^{th} class shall be given, is a curve of the n^{th} degree. For if the sum of the angles, made with the axis of x, $= 0$, the sum of the angles made with a line inclined at an angle $\frac{\theta}{n}$ to that axis is θ. Whatever be the fixed line or the angle, the locus will pass through the foci of the curve. This may appear paradoxical, since it follows hence that the sum of the angles made by the tangents from a focus, with any line, may be equal to any given quantity. The reason of this is, that two of these angles are those whose tangents are $\pm \sqrt{-1}$; these angles then are $\pm$ infinity (Art. 117); but the difference of two infinities may be any finite quantity. A line making any finite angle with a line through one of the circular points at infinity must be considered as coinciding with it (Art. 117); hence it follows that every focus of a curve is also a focus of its involute and evolute.

128. The tangential equation of the curve gives at once an important property of the perpendiculars let fall from the foci on any tangent. For let $\alpha\beta\gamma\delta$, &c. be the n foci, then the tangential equation must be of the form

$$\alpha\beta\gamma\delta \ldots \&c. = \omega\omega'\phi_{n-2},$$

where ϕ_{n-2} is a function of the degree $n-2$ in the tangential coordinates (for $\alpha\omega$, $\alpha\omega'$ are to be tangents to the curve, &c.).

Now for curves of the second class this at once gives the pro-

perty that the product of the perpendiculars from the two foci on any tangent is constant, since it was proved (Art. 10) that for $\omega\omega'$ we might substitute a constant.

For curves of the third class the equation is

$$\alpha\beta\gamma = k\xi,$$

where we replace $\omega\omega'$ by a constant, and where ξ is the point in which, as the equation shows, the three focal tangents intersect, which do not pass through ω, ω'. We learn then that the product of the three focal perpendiculars on any tangent to a curve of the third class, is in a constant ratio to the perpendicular on the same tangent from the point ξ.

For curves of the fourth class the equation is

$$\alpha\beta\gamma\delta = k\phi,$$

where ϕ is the conic section, which, as the equation shows, is touched by the eight focal tangents, which do not pass through ω, ω'. But if the foci of this conic be $\epsilon\zeta$, the equation may be put into the form

$$\alpha\beta\gamma\delta = k\epsilon\zeta + l,$$

the geometrical interpretation of which is obvious.

And so in general, the tangential equation may be interpreted, so as to give a relation of the first degree between the product of the n focal perpendiculars on any tangent, the product of $n - 2$ other perpendiculars, of $n - 4$ other, &c., and so on until we come either to a single perpendicular or to a constant term.

129. We shall in the next Chapter give an account of some of the focal properties of curves of the third degree, but little attention having hitherto been paid by geometers to the foci of curves of higher orders, we think it advisable to illustrate here, by a few examples, the use which may be made of them. The most interesting cases are those when two or more of the foci coincide, that is, when the curve passes through the circular points at infinity.

Ex. 1. Mention has already been made (Art. 124) of the *Cartesian oval*, or the locus of a point whose distances from two fixed points are connected by the relation $m\rho + n\rho' = d$. This evidently becomes an ellipse or hyperbola when $m = \pm n$, and a circle when d vanishes.

Let $A = 0$, $B = 0$ be the equations of the infinitely small circles whose centres are the two given fixed points; and the equation of the locus is

$$l\sqrt{A} + m\sqrt{B} = d,$$

or (*Conics*, Art. 282)

$$d^4 - 2d^2(l^2A + m^2B) + (l^2A - m^2B)^2 = 0. \qquad (1)$$

This equation may be written in the form

$$\left(l^2A - m^2B - \frac{l^2 + m^2}{l^2 - m^2}\, d^2\right)^2 + \frac{4l^2m^2}{l^2 - m^2}\,(A - B - d^4) = 0, \qquad (2)$$

which we may, for shortness, write $S^2 = b^3L$, where S represents a circle, and L a right line. This form of the equation shows at once that the line L is a double tangent (namely at the two points where it meets the circle S), and also that the two points are cusps, where the line at infinity meets the same circle. The curve, then, having two cusps, is only of the sixth class. Moreover, since the tangent at a cusp counts for three among the tangents which can be drawn to the curve from that point, it is easy to see that three of the foci of this curve must coincide. The triple focus is in fact the centre of the circle S, for the imaginary asymptotes to that circle are also asymptotes to the curve. Beside this, then, there are three other foci; two of them are the fixed points A and B; for the equation may be written in the form

$$l^2A(l^2A - 2m^2B - 2d^2) + (d^2 - m^2B)^2 = 0,$$

showing that the two imaginary lines represented by A are tangents to the curve. Similarly for B. Now it is natural to suppose that the third focus (C) is not different in character from the other two, and that the equation of the curve may also be written in the form

$$m'\sqrt{A} + n'\sqrt{C} = d',$$

or (eliminating the constant by the help of the first form of the equation) in the form

$$l\sqrt{A} + m\sqrt{B} + n\sqrt{C} = 0.$$

This is, in fact, what M. Chasles has proved, though apparently he was not led to anticipate his result by considerations such as are here suggested.

Let three points, a, b, c, be on a right line, and take any other point P; expressing the cosines of the supplemental angles at b,

in terms of the sides of the triangles to which they respectively belong, we have

$$\frac{ab^2 + bP^2 - aP^2}{ab} + \frac{bc^2 + bP^2 - cP^2}{bc} = 0,$$

or
$$ab \,.\, bc \,.\, ca + ac \,.\, bP^2 = bc \,.\, aP^2 + ab \,.\, cP^2. \qquad (3)$$

Thus we see that if A, B, C represent the infinitely small circles whose centres are three points on a right line, these are connected by an equation of the form

$$\lambda A + \mu B + \nu C = d^2.$$

Now M. Chasles's theorem will be proved if we can show that λ, μ, ν can be so determined that, on substituting this value of d^2, the equation (1) will be transformed into one of the form

$$(l_1^2 A - m_1^2 B)^2 - 2n_1^2 C (l_1^2 A + m_1^2 B) + n_1^4 C^2 = 0$$

But the result of the substitution in equation (1) is

$$\nu^2 C^2 - 2\nu C \{(l^2 - \lambda) A + (m^2 - \mu) B\} + (l^2 - \lambda)^2 A^2 + (m^2 - \mu)^2 B^2$$
$$- 2(l^2 m^2 + l^2 \mu + m^2 \lambda - \lambda\mu) AB = 0.$$

This will be of the required form if

$$(l^2 - \lambda)(m^2 - \mu) = l^2 m^2 + l^2 \mu + m^2 \lambda - \lambda\mu,$$

or
$$\frac{l^2}{\lambda} + \frac{m^2}{\mu} = 1.$$

If we attend to the actual values of λ, μ, as inferred from equation (3), we can deduce that the curve

$$l\sqrt{A} + m\sqrt{B} = d$$

has a third focus C on the same line with the other two, whose position is determined by the equation

$$ab\,(l^2 . ac + m^2 . bc) = d^2.$$

Every curve of the fourth degree, having the two circular points at infinity for cusps, is a Cartesian oval. For, to be given that a point is a cusp is equivalent to four conditions; hence curves of this class are determined by six new conditions. To be given that a point is a focus is equivalent to two conditions, since we are given two tangents drawn through that point. The equation, therefore, $l\sqrt{A} + m\sqrt{B} = d$, containing two expressed constants and four implied conditions, is the most general equation of its class.

Ex. 2. Let us next take the class of curves of which we have just seen that the Cartesian oval is a particular case,

$$l\sqrt{A} + m\sqrt{B} + n\sqrt{C} = 0,$$

or $\quad l^4A^2 + m^4B^2 + n^4C^2 - 2l^2m^2AB - 2l^2n^2AC - 2m^2n^2BC = 0,$

a curve of the fourth degree, of which the three given points are obviously foci, since the equation can be thrown into the form

$$l^2A(l^2A - 2m^2B - 2n^2C) + (m^2B - n^2C)^2 = 0.$$

On substituting for A, B, C their values in x and y co-ordinates, the equation will assume the form

$$(x^2 + y^2)^2 + u_1(x^2 + y^2) + u_2 = 0;$$

showing that the two circular points at infinity are double points on the curve. The curve is therefore of the eighth class, and has two double foci, since at each of these circular points there are two tangents, the intersection of either of which with the corresponding tangent gives rise to a double focus. There are consequently four other foci, and therefore one in addition to the three given points.

The curve is only of the third degree if $l \pm m \pm n = 0$ (or rather the locus breaks up into a curve of the third degree, together with the line at infinity), for the coefficient of $(x^2+y^2)^2$ is

$$l^4 + m^4 + n^4 - 2l^2m^2 - 2m^2n^2 - 2n^2l^2$$
$$= -(l+m+n)(l+m-n)(l-m+n)(m+n-l).$$

In this case, however, there are still four foci; for the two points at infinity are then only ordinary points on the curve, which is of the sixth class, and has one double and four single foci.

In either case it is natural to conjecture, and we now proceed to prove, that the fourth focus possesses the same property as the others; or that the equation of the curve may be also thrown into the form $\qquad l\sqrt{A} + m\sqrt{B} + n\sqrt{D} = 0.$

First. Let the three given points lie on a right line. It is proved, as in the last example, that the distances of any assumed point P from four points, a, b, c, d, in a right line, are connected by the relation

$$\frac{A}{ab.ac.ad} + \frac{B}{ba.bc.bd} + \frac{C}{ca.cb.cd} + \frac{D}{da.db.dc} = 0,$$

where the line ba, for example, is equal in length, but opposite in sign to the line ab.

But it is proved precisely as in the last example that if in the equation
$$l\sqrt{A} + m\sqrt{B} + n\sqrt{C}$$
we substitute
$$n^2 C = \lambda A + \mu B + \nu D,$$
it will be transformed into the form
$$l'\sqrt{A} + m'\sqrt{B} + n'\sqrt{D},$$
provided we have
$$\frac{l^2}{\lambda} + \frac{m^2}{\mu} = 1;$$
and, attending to the values given for $\lambda\mu\nu$, we see that the curve will have a fourth focus D situated on the same line as the other three, and whose position is determined by the equation
$$l^2.ab.ac.ad + m^2.ba.bc.bd + n^2.ca.cb.cd = 0.$$

The fourth focus goes off to infinity, and the equation represents a Cartesian oval when the algebraic sum vanishes of the products of the square of each coefficient by the rectangle under the distances from the corresponding focus to the other two.

Ex. 3. Or generally, let the three given points be situated in any manner. The foregoing proof will still apply if we can find a fourth point d, such that the distances of any assumed point P from the four points shall be connected by a relation of the form $D = \lambda A + \mu B + \nu C$. But (*Conics*, p. 102) this will be the case if d be taken on the circumference of the circle through the three given points, since then
$$bcd.A + cda.B + dab.C + abc.D = 0,$$
where abc, &c. is the area of the triangle formed by these points, and where the signs to be given to the triangles are alternately positive and negative as we proceed in order to round the circle.

The condition
$$\frac{l^2}{\lambda} + \frac{m^2}{\mu} = 1$$
gives us the following equation to determine the fourth point d when a, b, c, and when l, m, n are given,
$$\frac{l^2}{bcd} + \frac{m^2}{acd} + \frac{n^2}{bad} = 0.$$

Let the distances of d from bc, ca, ab, be α, β, γ, and this equation is
$$\frac{l^2\beta\gamma}{bc} + \frac{m^2\gamma\alpha}{ca} + \frac{n^2\alpha\beta}{ab} = 0,$$
the equation of a conic through the given points, whose fourth point of intersection with the circle through the given points gives the point d required. This conic coincides with the circle itself when $l : m : n = bc : ca : ab$; but in this case the locus reduces to the square of that circle, as is evident by Ptolemy's theorem. *Hence all curves of the third degree whose highest terms are divisible by $x^2 + y^2$; and all curves of the fourth degree whose equations are of the form*
$$(x^2 + y^2)^2 + u_1(x^2 + y^2) + u_2 = 0,$$
have four foci lying on a circle, any three of which have the property, in the first case, $lr + ms = (l + m)t$; in the second case, $lr + ms + nt = 0$, where r, s, t are the distances of any point on the curve from the three foci.

We shall enter into more detail on this subject in the next Chapter.

Ex. 4. Equations of the form just mentioned include many well-known curves of the fourth degree. For example, the *ovals of Cassini* are the locus of the vertex of a triangle of which the base and rectangle under sides are given.

Taking the middle point of the base $2a$ for origin, the given rectangle being ab, the equation is
$$(x^2 + y^2 + a^2 - 2ax)(x^2 + y^2 + a^2 + 2ax) = a^2b^2,$$
or
$$(x^2 + y^2)^2 + 2a^2(x^2 + y^2) - 4a^2x^2 + a^4 - a^2b^2 = 0,$$
an equation of the class in question. To find its foci, apply the condition that $x + y\sqrt{-1} = p$ should touch the curve, which is the condition that the following equation should have equal roots:
$$p^4(x - y\sqrt{-1})^2 + 2a^2p^2(x^2 + y^2) - 4a^2p^2x^2 + (a^4 - a^2b^2)(x + y\sqrt{-1})^2 = 0.$$
This condition is
$$(p^4 - 2a^2p^2 + a^4 - a^2b^2)(-p^4 + 2a^2p^2 - a^4 + a^2b^2) + (p^4 - a^4 + a^2b^2)^2 = 0,$$
which reduces to
$$(p^2 - a^2)(p^2 - a^2 + b^2) = 0.$$

The curve, then, has not only the extremities of the base of the triangle for foci, but also the feet of the perpendiculars let fall

from the vertices of the two right-angled triangles which belong to the locus.

130. In these examples attention has only been paid to the properties of the single foci of the curves discussed; those of the multiple foci would probably also repay examination.

Thus the Cartesian ovals have a triple focus, through which if any chord be drawn, the sum of the segments on one side is equal to that of the segments on the other. And the form (2) of the equation shows that the fourth power of the tangent drawn from any point of the curve to the circle having this point as centre, and passing through the points of contact of the double tangent, is in a constant ratio to the distance of the same point on the curve, from the double tangent.

The equation $lr + ms + nt = 0$ admits of being thrown into the form $SS' = k^3L$, where S, S′ are circles, of which the double foci are the centres, and where L represents a right line. The geometrical interpretation of this equation is obvious.

In the ovals of Cassini, the circles S, S′ reduce to points, and the line L goes off to infinity. They are the form which the general curve assumes when the two double foci coincide with two of the single foci, and when the two circular points at infinity are double points, whose tangents meet the curve in four consecutive points.*

SECT. VIII.—TRACING OF CURVES.

131. It may be proper to give some examples of the method of tracing the figure of a curve from its equation. If we give any

* M. Plücker, in an early volume of Crelle's Journal, has given the definition of foci of higher curves which I have employed in this section. He has not, however, as far as I am aware, investigated the properties of these points. The advantage of considering these foci had occurred to me, in ignorance of M. Plücker's suggestion, and I proposed to Dr. Hart to join me in an examination of their properties, for curves of the third and fourth orders. Almost all the results of our investigation are due to Dr. Hart. In particular, I am indebted to him for the extension of M. Chasles' theorem with regard to Cartesian ovals, contained in Art. 129, Ex. 3; and nearly all the materials for the section in the next Chapter, on the focal properties of curves of the third order, have been contributed by him. No doubt many important properties of the foci of higher curves remain still to be discovered.

value (a) to either of the variables x, the resulting numerical equation can be solved (at least approximately) for y, and will determine the points in which the line $x = a$ meets the curve. By repeating this process for different values of x, as at *Conics*, p. 12, we can obtain a number of points on the curve; and by drawing a line freely through them, can obtain a good idea of its figure. By taking notice what values of x render any of the values of y imaginary, we can perceive the existence of ovals, or can observe whether the curve is limited in any direction; and we have already shown (Art. 43) how to find whether the curve has infinite branches, and how to determine its asymptotes. We have also shown how to find its multiple points and points of inflexion. The value of $\frac{dy}{dx}$ at any point gives the direction of the tangent at that point (Art. 39); and if we examine for what points $\frac{dy}{dx} = 0$, or $= \infty$, we shall have the points at which the course of the curve is parallel or perpendicular to the axis of x.

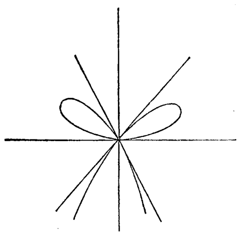
In practice we must, of course, take advantage of any simplifications which the equation of the curve suggests. Thus, if we consider a series of lines parallel to one of the asymptotes (or a series of lines passing through a point on the curve), the equation which determines the other points in which any of them meets the curve, is of a degree one lower than the degree of the curve. If the equation shows that the curve has a double or other multiple point, it is advantageous to consider a series of lines drawn through this point, since then the equation in question will lose two or more dimensions.

It seems unnecessary to add more than one or two examples to those which incidentally occur in the course of these pages. We refer the reader who may wish for further illustration to Gregory's Examples, chap. xi.; or, if still unsatisfied, to the source whence all later writers on the subject have drawn largely, Cramer's Introduction to the Analysis of Curves.

Ex. 1. $x^4 - ax^2y + by^3 = 0$ (see p. 41).

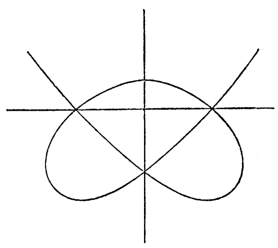
Here, the origin being a triple point, it is advantageous to consider a series of lines drawn through it. Substituting $y = mx$, we find

$x = m(a - bm^2)$, a function which increases from 0, when $m = 0$, to a maximum value when $a - 3bm^2 = 0$; decreases then, and vanishes when $a - bm^2 = 0$ and has an indefinitely increasing negative value as m increases further. The curve is manifestly symmetrical on both sides of the axis of y. Hence the figure is that here represented.



Ex. 2. $(x^2 - a^2)^2 = ay^2(3a + 2y)$, (see p. 42).

Hence $x^2 = a^2 \pm \sqrt{\{ay^2(3a + 2y)\}}$. The curve is plainly symmetrical on both sides of the axis of y. It has on each side two branches, corresponding to the two signs we may give the radical. The two branches intersect when $y = 0$, and accordingly we have seen that there are on the axis of x two double points at the distance $x = \pm a$. As y increases positively, the radical increases indefinitely; hence the value of x, corresponding to the one branch, increases indefinitely; that corresponding to the other decreases, until we come to the value of y, corresponding to the single positive root of the equation $2ay^3 + 3a^2y^2 = a^4$, $(2y = a)$, beyond which this branch can extend no higher. For negative values of y, the radical increases to a maximum value when $y + a = 0$; the one pair of branches then intersect in a double point on the axis of y, and the other pair is at its furthest distance from that axis. Evidently neither branch can proceed lower than the value $3a + 2y = 0$. Hence the shape of the curve is that represented in the figure.



CHAPTER III.

CURVES OF THE THIRD DEGREE.

SECT. I.—PRINCIPAL FORMS OF THE EQUATION OF THE THIRD DEGREE.

132. THE most important division of curves of the third degree is made with reference to the class of the curve. We have seen (Art. 70) that if the curve have no multiple point, it will be of the sixth class, and (Art. 34) that the curve may have one double point, but not more. Cubics* may then be subdivided into (A) curves of the sixth class; (B) curves of the fourth class, (a) having a node, (b) having a conjugate point; and (C) curves having a cusp, which must be of the third class.

This classification is based on fundamental properties of the curve, and not merely on accidental varieties of shape, and is accordingly adopted in the following pages. We might, however, have classified curves of the third, like those of the second degree, with reference to the number of their infinite branches; a distinction, however, which, being lost in projection, cannot be considered as founded on essential differences of the curves. The line at infinity meets every cubic in three points, which may either be (1) all real and distinct, (2) one real and two imaginary, (3) one real and two coincident, (4) all three coincident. In the first case the curve will have three, in the second case one, real asymptote, and to each asymptote will (as in the conic sections) correspond a pair of infinite branches; in the third case, it will have a pair of infinite branches touching a real asymptote, and, besides, a pair of branches not approaching to contact with any finite asymptote, but touched by the line at infinity. Infinite

* To escape the irksome repetition of the periphrasis, "curve of the third degree," I have taken the liberty to extend the use of the term "cubic," applied in algebra to equations of the third degree.

branches of the latter kind we shall call *parabolic*, those of the former, *hyperbolic*. In the fourth case, the line at infinity is a tangent at a point of inflexion, and the curve has one pair of parabolic infinite branches. Such curves, having no finite asymptote, may be called parabolæ of the third degree.

There may be curves of each of these species belonging to each of our three primary divisions: but besides, there are others which can only contain curves of the third or fourth class, for (5) the line at infinity might pass through a double point and an ordinary point; this only differs from species (1) in that two of the asymptotes are parallel, being the tangents at the infinitely distant double point; (6) it might pass through a conjugate point and an ordinary point; (7) through a cusp and an ordinary point; in this case the two parallel asymptotes of species (5) unite; (8) it might touch at a double point; in which case the curve has one parabolic pair of infinite branches, and also a pair touching an asymptote parallel to the direction of the point at infinity on those branches. (9) Lastly, it may touch at a cusp, when we have a single parabolic pair of infinite branches.

These constitute the principal varieties of shape arising from the nature of the points at infinity on the curve. A more minute discussion of the different possible varieties of shape will be better understood after we have first explained some of the principal properties of curves of the third degree.*

133. Although we commenced our discussion of curves of the second degree by an examination of the general equation, yet, as in the present case such an examination has some difficulty, and not much interest, for beginners, we purpose to reserve it for the last, and shall commence by noting (as at *Conics*, Chap. VII.) some of the principal forms which the general equation may as-

* The first attempt to enumerate the different species of curves of the third order was made by Newton, "Enumeratio linearum tertii ordinis." A few species, overlooked by Newton, were pointed out by his commentator, Stirling. The most recent classification has been made by Plücker, who has investigated this subject with great minuteness, and in his "System der Analytischen Geometrie" has counted no fewer than 219 different species of curves of the third degree. As many of the species treated by Plücker as distinct do not appear to me to differ essentially, either in form or in properties, I shall content myself with a much smaller number of species.

sume; by which means we shall have an opportunity of introducing to the reader several important properties. We commence with binomial equations of the form $S + kS' = 0$. We have already (Art. 22) made the reader acquainted with the theorem that the nine points of intersection of S and S′ are such that, eight of them being given, the ninth can be found; and that if three of them lie on a right line, the remaining six will lie on a conic, and *vice versâ*.

(1.) $$ACE - kBDF = 0.$$

This equation contains thirteen constants (two being implicitly contained in the equation of every right line), therefore every equation of the third degree may be reduced to this form in an infinity of ways. The same thing appears geometrically, for if we draw any two lines A, C, and draw B, D, F, to join in pairs the points where these lines meet the curve, we see, by the theorem just cited, that the three points where B, D, F meet the curve again, lie on a right line E.

We may also infer from this equation, that *if a conic be described through four points on a curve of the third degree, the line joining the two points where the conic meets the curve again, will pass through a fixed point on the curve*, which we shall call the point *opposite* to the four fixed points. For let the equation of the conic be $AC = \lambda BD$; introducing this into the equation $ACE = BDF$, we have $\lambda E = F$, a right line passing through the fixed point EF. The point opposite to four given points is simply constructed as follows: "Let the line joining one pair of points meet the curve in e, and the line joining the other pair in f, then ef meets the curve again in the point required."

The form of the equation $ACE = BDF$, gives us at once the locus of a point such that the product of its distances from three fixed lines may be in a constant ratio to the product of its distances from three other fixed lines.

134. (2) If the lines B, D coincide, the equation becomes

$$ACE - B^2F = 0,$$

containing implicitly eleven constants, and therefore one to which every equation of the third degree may be reduced in many different ways. B may be taken for any line meeting the curve in

three real points, then we have seen (Art. 42) that A, C, E are the tangents at these points, and it appears now that *if at three points of the curve which are in a right line, tangents be drawn, the three points where the tangents meet the curve again, lie in a right line* F.

The preceding equation may be written in a form which will include the case where B meets the curve in imaginary points,

$$A(C^2 \pm E^2) - B^2F = 0.$$

The theorem just given may be otherwise stated thus: "Let A, B, C be three points of the curve on a right line; we know that from each of these, four tangents can be drawn to the curve (Art. 79), let their points of contact be a_1, a_2, a_3, a_4, &c., then the line joining a, b will pass through c, one of the points of contact of tangents from C." For the tangent at the point where a, b meets the curve must, by this theorem, pass through C.

Let us now suppose that the line AB is the tangent to the curve at A, the points A, B will coincide, and we see that the line a_1c_1 must pass through one of the *other* points of contact of tangents from A, suppose a_2. In like manner, a_3c_1 must pass through a_4. We have then the theorem due to Mac Laurin: *If we complete the quadrilateral formed by $a_1a_2a_3a_4$, the points of contact of tangents from any point on the curve* A, *then the intersection of diagonals, and the intersections of opposite sides, $c_1c_2c_3$ will be also points on the curve, and the tangents at these points and the tangent at* A *will all meet the curve in the same point* C.

135. An extension of this theorem is, that if $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, be the two sets of tangents from two points on the curve A, B, then the lines joining A_1B_2, A_2B_1, intersect on the curve.

The line joining the points of contact of A_1B_1 passes through the point of contact of C_1, one of the tangents from the point C, where AB meets the curve again; and let us suppose that they have been so numbered that the line joining the contacts of A_2B_2 passes through *the same* point of contact.

Then the equation of the curve is of the form

$$A_1B_1C_1 = D^2F,$$

where D is the chord of contact, and F the line ABC. It must also be of the form

$$A_2B_2C_1 = D_1^2F.$$

Hence we must have identically

$$C_1(A_1B_1 - A_2B_2) = F.(D^2 - D_1^2).$$

The right-hand side of the equation denotes three right lines, therefore the left-hand side must denote the same right lines; hence one of the factors of $A_1B_1 - A_2B_2$ is F, and the other, which joins A_1B_2, A_2B_1, must be $D \pm D_1$, C_1 being $D \mp D_1$. We see then that the latter two lines and the two chords D, D_1 form a harmonic pencil, whose vertex is the point of contact of C_1.

In the case where the points A, B coincide, the line joining A to the point of contact of C_1, C_1 itself, and the two chords D, D′ form a harmonic pencil.

136. Hence can be deduced another theorem of Mac Laurin's. Any line drawn through a point A on a cubic is cut harmonically in the two points β, γ, where it meets the cubic again, and the two points δ, δ', where it meets a pair of chords joining the points of contact of tangents from A. Let the line meet the tangent C_1 in the point e, then, since it meets A_1 and B_1 at A, by Art. 60,

$$\frac{1}{\delta A} + \frac{1}{\delta\beta} + \frac{1}{\delta\gamma} = \frac{2}{\delta A} + \frac{1}{\delta e},$$

or

$$\frac{1}{\delta\beta} + \frac{1}{\delta\gamma} = \frac{1}{\delta A} + \frac{1}{\delta e}.$$

But, by the last Article, $\delta\delta'$ is a harmonic mean between δA and δe, therefore also between $\delta\beta$ and $\delta\gamma$. Q. E. D.

When the curve has a double point, only two tangents can be drawn to the curve; but the theorem of this Article will be still true, if for the chord D′ we substitute the line joining the double point to the point where the chord D meets the curve again.

137. Equation (2) also includes the following

$$(3)\ \ ACE - k^2F = 0, \text{ or } A(C^2 \pm E^2) = k^2F,$$

an equation containing nine constants, and therefore one to which every curve of the third degree must be reducible. A, C, E are evidently the three asymptotes, and we have the theorems, "*The asymptotes, if all real, meet the curve again in three finite points, which*

lie in a right line F, and *the product of the distances of any point of the curve from the three asymptotes is in a constant ratio to its distance from the line* F.

In equation (2) let BDF coincide, and the equation becomes

$$(4)\ \ \mathrm{ACE} - \mathrm{B}^3 = 0,$$

an equation also containing nine constants. A, C, E are (Art. 42) tangents at points of inflexion, which lie on a right line B, and we have the theorem proved already, Art. 49, *the line joining two points of inflexion must pass through a third.* And again, *the cube of the distance of any point of the curve from this line is in a constant ratio to the product of its distances from the three tangents at the points of inflexion.*

We give here another form, on account of its connexion with points of inflexion.

$$(5.)\ \ a\mathrm{A}^3 + b\mathrm{B}^3 + c\mathrm{C}^3 - 3d\mathrm{ABC} = 0,$$

an equation involving nine constants, three expressed, and six implied, and therefore one to which every cubic equation can be reduced. By including a multiplier in the equations of each of the lines ABC we can reduce the number of expressed constants, and write the equation

$$\mathrm{A}^3 + \mathrm{B}^3 + \mathrm{C}^3 - 3d\mathrm{ABC} = 0.$$

Each of the lines A, B, C passes through three points of inflexion, for let θ be one of the imaginary cube roots of unity, and we may write the equation

$$(d\mathrm{A} + \mathrm{B} + \mathrm{C})\ (d\mathrm{A} + \theta\mathrm{B} + \theta^2\mathrm{C})\ (d\mathrm{A} + \theta^2\mathrm{B} + \theta\mathrm{C}) = (d^3 - 1)\mathrm{A}^3;$$

therefore A passes through three points of inflexion. Similarly for B and C.

If in equation (4) B pass through the intersection of A and C, the equation takes the form

$$(6.)\ \ \mathrm{ACE} = (\mathrm{A} + k\mathrm{C})^3,$$

a form containing but eight constants, and therefore one to which every cubic cannot be reduced. In fact the curve has AC for a double point, at which A and C are tangents: E is a tangent at a point of inflexion which lies on the line $\mathrm{A} + k\mathrm{C}$.

Let A, C coincide, and the equation becomes

$$(7.)\ \ \mathrm{A}^2\mathrm{E} = \mathrm{B}^3,$$

containing seven constants only, and therefore one to which the

general equation cannot be reduced unless two conditions be fulfilled. And the same appears from Art. 101, since the equation denotes a curve having a cusp at which A is a tangent. E is a tangent at a point of inflexion lying on the line B.

138. The most general trinomial equation of the third degree is
$$S + kS_{,} + lS_{,,} = 0.$$
It may be taken as the most general equation of such a curve passing through seven given points, if S, $S_{,}$, $S_{,,}$ be any particular curves of the system. For the equation contains explicitly two constants, which can be determined so that the curve shall pass through any other two given points, since we have two linear equations to determine k and l, if we substitute in the equation the co-ordinates of the two new given points.

139. In like manner, the most general equation of a cubic through six given points, must consist of four terms; and we may take for each of these the product of three right lines, each passing through two of them. Thus, the six points being a, b, c, d, e, f, and ab denoting the equation of the line joining a, b, one form of the equation of the required cubic is
$$ab.cd.ef + \lambda ac.be.df + \mu ad.bf.ce + \nu ae.bd.cf = 0.$$
Since this equation contains three indeterminates, every other cubic through the six points (for example, $af.bc.de$) must be capable of being expressed in the above form, and the preceding equation would gain no generality if we were to add to it a term $\pi.af.bc.de$, since this itself must be the sum of the preceding four terms multiplied each by some factor.

In precisely the same manner as (*Conics*, p. 218) we derived the anharmonic property of the points of a conic from the equation $ab.cd = k.ac.bd$, we can derive from the equation just written the following, which is the extension of the anharmonic theorem to curves of the third degree: "If six given points on such a curve be joined to any seventh, and if any transversal meet this pencil in points a, b, c, d, e, f, then the relation holds
$$ab.cd.ef + \lambda ac.be.df + \mu ad.bf.ce + \nu ae.bd.cf = 0,$$
where λ, μ, ν are constants, whose value is the same for each particular curve through the six points." The reader can easily con-

ceive the number of particular theorems which may be derived from this (as in *Conics*, Art. 315), by examining the cases where some of the points are at an infinite distance.

140. We saw (Art. 36) that to be given a double point was equivalent to three conditions. If then we have a double point and five other points, one more condition will determine the curve, which may, therefore, be expressed by an equation of the form $S - kS' = 0$, where S, S' are two particular curves of the system. We may write it in the form

$$(oabcd)\,oe - \lambda(oabce)\,od = 0,$$

where $(oabcd)$ denotes the conic through the double point o and the four points $abcd$.

In like manner we may write the equation of the cubic through the double point and four other points

$$oa.ob.cd + \lambda.ob.oc.ad + \mu.oc.oa.bd = 0;$$

and, as in the last Article, the same relation holds between the intercepts on any transversal by the line joining these points to any point of the curve.

141. By the help of the same method (*Conics*, p. 218) of expressing the anharmonic ratio of a pencil in terms of the perpendiculars let fall from its vertex on the sides of any quadrilateral whose vertices lie each on a leg of the pencil, we can find the locus of the common vertex of two pencils, whose anharmonic ratio is the same, and whose legs pass through fixed points, two of the fixed points being common to both pencils. For if ab denote the equation of the line joining the points ab, we get an equation of the form

$$\frac{ao.bp}{ab.po} = \frac{co.dp}{cd.op},$$

or

$$ao.bp.cd = ab.co.dp,$$

an equation of the class discussed in Art. 133.

When op are the two circular points at infinity, this gives us (*Conics*, p. 315) the locus of the common vertex of two triangles whose bases are given and vertical angles are equal, and we see that it is a curve of the third degree passing through those circular points.

If the difference of the vertical angles were given, this would

be equivalent (*Conics*, p. 315) to the ratio of two anharmonic functions, and we should be led to an equation of the form

$$\frac{ao \, . \, bp}{ap \, . \, bo} = k \frac{co \, . \, dp}{cp \, . \, do},$$

which represents a curve of the fourth degree, having the two circular points for double points.

SECT. II.—POINTS OF INFLEXION.

142. We add next some important theorems relating to the points of inflexion of curves of the third degree. We commence with the following problem:

"*Radii vectores through a point* O *on the curve meet the curve again in* A, B: *to find the locus of the harmonic means of* OA, OB."

Take the point O for origin, and let the equation of the curve be

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Ky = 0.$$

Transforming to polar co-ordinates, we find for the harmonic mean,

$$\frac{1}{\rho'} + \frac{1}{\rho''} = \frac{2}{\rho} = - \frac{E\cos^2\theta + F\cos\theta\sin\theta + G\sin^2\theta}{H\cos\theta + K\sin\theta},$$

and therefore the equation of the locus required is

$$Ex^2 + Fxy + Gy^2 + 2(Hx + Ky) = 0,$$

but this is the polar conic of the point O (Art. 57). Hence *every chord through* O *is cut harmonically by the curve and the polar conic of* O.

Now we have seen (Art. 84) that there are just two cases in which the polar conic of a point on the curve breaks up into two right lines. First, if O be a double point, H and K = 0, and the polar conic reduces to the two tangents at O. But this case does not apply to the present problem, since a line through a double point would only meet the curve in one other point. But secondly, it will also break up into two right lines if O be a point of inflexion. Taking, for simplicity, the tangent for the axis of x, we shall have H and E = 0, and the equation of the locus becomes

$$y(Fx + Gy + 2K) = 0;$$

and since the tangent y, passing through O, is plainly irrelevant to the question, we learn that, *if radii vectores be drawn through a*

point of inflexion, the locus of harmonic means will be a right line. And, conversely, *if the locus of harmonic means be a right line, the point* O *is a point of inflexion.* We shall call this line the harmonic polar of the point O, to distinguish it from the ordinary polar, which is the tangent at the point. The theorem just given is due to Mac Laurin (De Linearum Geometricarum Proprietatibus Generalibus, sect. iii. prop. 9).

143. The point O possesses, with regard to the harmonic polar, properties precisely analogous to those of poles and polars in the conic sections. Thus if two lines be drawn through O, and their extremities be joined directly and transversely, the joining lines must intersect on the harmonic polar. This is an immediate consequence of the harmonic properties of a quadrilateral.

Hence again, as a particular case of the last, tangents at the extremities of any radius vector through O must meet on the harmonic polar.

The harmonic polar must pass through the points of contact of tangents which can be drawn through O, for, since $OR'RR''$ is cut harmonically, if R' coincide with R'', it must coincide with R. Hence through a point of inflexion but three tangents can be drawn, and their points of contact lie on a right line.

If the curve have a double point, it is proved, in precisely the same way, that it must lie on the harmonic polar.

The first theorem of this Article may be otherwise stated thus: if three points $A'B'C'$ lie on a right line, and the lines joining O to them meet the curve again in $A''B''C''$, these will also lie on a right line, and the two lines will meet the harmonic polar in the same point. If now we suppose A, B, C to coincide, we arrive again at the theorem that the line joining two points of inflexion must pass through a third, and that the tangents at any two meet on the harmonic polar of the remaining one.

144. We have seen that every curve of the third degree has, at least, one point of inflexion; if, therefore, the tangent at this point be projected to infinity, the curve will be projected into one of the species (4), Art. 132. And since the point O is at infinity, every chord through it must be bisected by its harmonic polar. We see, then, that *every parabola of the third degree has a diameter*

which bisects all chords parallel to a certain line. Taking this diameter for axis of x, since the values of the ordinate must be equal and opposite for each value of x, the equation of the curve must be of the form

$$y^2 = Ax^3 + Bx^2 + Cx + D,$$

which we have discussed already (Art. 32). *Every curve of the third degree can therefore be projected into one of the five parabolæ included in equations of this form* (see Art. 149).

145. Instead of projecting the point of inflexion to infinity, we might so project its harmonic polar: we see then (as at *Conics*, p. 270) that the point of inflexion would become a centre, every chord through which would be bisected. The equation of the curve referred to the centre as origin is of the form (Art. 55),

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + y = 0.$$

The species of central curves are distinguished by the nature of the points of the curve at infinity, according as the curve has (1) three real asymptotes, (2) one real asymptote, and two imaginary, (3) or has a node, (4) or conjugate point, (5) or a cusp at infinity. For we have proved that if the curve have a double point, it must be on the harmonic polar. *Every curve then of the third degree is capable of being projected into one of these five central curves.**

146. *If through any point of inflexion* O, *there be drawn three right lines meeting the curve in* A_1, A_2, B_1, B_2, C_1, C_2, *then every curve of the third degree through the seven points* $OA_1A_2B_1B_2C_1C_2$ *will have* O *for a point of inflexion.* For let the three lines meet the harmonic polar in A, B, C, then these points are also common to the loci of harmonic means of the point O, with regard to all curves through the seven points. This locus, then, which would in general be a conic, must, since these three points of it are in a right line, be for all these curves this same right line; and therefore (Art. 142) the point O must be a point of inflexion.

147. We have seen (p. 71) that the points of inflexion of a

* The theorem of the last Article is Newton's. The proof here given, and the extension of this Article, are due to M. Chasles.

curve of the third degree are the intersections of the curve U with the curve H, which is also a curve of the third degree. *Every curve of the third degree has therefore, in general, nine points of inflexion*, only three of which, however, are real (see Art. 49). Since, also, we have proved that the line joining two points of inflexion must pass through a third, through each point of inflexion can be drawn four lines, which will contain the other eight points. It follows then, as a particular case of the last Article, that *any curve of the third degree, described through the nine points of inflexion, will have these points for points of inflexion.**

148. Of the lines which each contain three points of inflexion, since four pass through each point of inflexion, there must be in all $\frac{4 \times 9}{3} = 12$.

If we attempt to form a scheme of these lines, it will be found that it can only differ in notation from the following:

123,	468,	579.
145,	269,	378.
167,	285,	349.
189,	365,	247.

Hence it will follow that any cubic passing through any seven of the points of inflexion will have one of these for a point of inflexion; for, take any seven (say the first seven), and it will appear from the above table that they lie on three right lines (123, 145, 167), intersecting in a common point on the curve, and therefore, by the last Article, that common point (1) is a point of inflexion on them all.

From the manner in which these lines have been written, it appears that they may be divided into four sets of three lines, each set passing through all the nine points; or that, if we form the equation $U + \lambda H$, there are four values of λ, for which the equation reduces itself to a system of three right lines. For a direct proof of this, see the last section of this Chapter. The exis-

* This theorem is due to M. Hesse (see p. 75). The method of proof here adopted is Dr. Hart's.

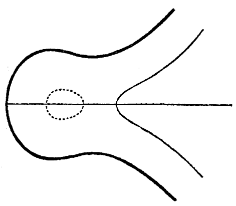
tence of at least one set of three lines passing through the nine points of inflexion appears from Art. 137 (5).

SECT. III.—ON THE FIGURES OF CURVES OF THE THIRD DEGREE.

149. We shall take advantage of the theorem of Art. 144, to make a classification of the different possible varieties of shape which can occur in curves of the third degree. It is plain that we have only to consider (as in Art. 32) the different varieties of shape which cubics can possess when the line at infinity touches at a point of inflexion, and then to examine the effect produced on the figure by projection, according to the different possible positions of the line projected to infinity. The equation

$$y^2 = Ax^3 + Bx^2 + Cx + D$$

includes two species of the sixth class; one with a real oval, when the right-hand side can be decomposed into three real and unequal factors (discussed in Art. 32); and the other when two of these factors (and consequently the oval) are imaginary, and when the curve consists only of a pair of infinite branches. The equation includes two species of the fourth class; one with a node, and one with a conjugate point; and one species of the third class. We reckon, therefore, in all, five parabolæ of the third degree. The figures given (p. 30) sufficiently illustrate the last three species. We give a new figure for the first two species, since the figure at p. 29, having been drawn merely with the view of exhibiting the existence of an oval, does not correctly represent the form of the infinite branch. In fact, the form of the equation shows that the point of contact of the line at infinity with the curve is on the line x, unlike the common parabola $y^2 = px$, which is touched at infinity by the line y; the infinite branches, therefore, of the cubic ultimately tend to become parallel to the axis of y, and not to the axis of x; and there must be a finite point of inflexion on each side of the diameter, where the curve changes from being concave to being convex towards the axis of x. The oval and the inner infinite branch represent the first species; the second species will have a form



either resembling the infinite branch of the first species, or else such as is given in the outer curve, according as the equation $\frac{dy}{dx} = 0$ gives imaginary or real values for x. The same equation will show that the highest point of the oval is on the side remote from the infinite branch, and that the oval approaches more nearly to the form of an ellipse the greater its distance from the infinite part.

It is, of course, to be understood that the ordinates are not necessarily perpendicular to the diameter.

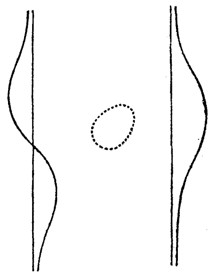
In this and the following figures I have dotted the oval and its projections, which we have only to suppose removed in order to obtain an idea of the second species and its projections. By supposing the oval to shrink into a point, the same figure may serve to represent the species which have a conjugate point.

150. It is plain, in general, that every line must meet any closed figure in an even number of real points, and in the case when a cubic has an oval, that every line which meets the oval part of the curve once must meet it once again, and not oftener; since, when a line crosses to the inside of the oval, it must cross it again to come out, and cannot meet the oval in four points. Every line, therefore, must meet the infinite part of the curve once. It follows that no tangent to the curve can meet the oval again, and, therefore, that none of the points of inflexion can lie on the oval.

It may be seen, as at *Conics*, p. 303, that if, in projecting the figure, the line which is sent off to infinity do not meet the oval, the projection of the oval will still be a closed figure; if the line should touch the oval, the projection will be a pair of parabolic infinite branches; if the line cut the oval, the projection will be two pairs of hyperbolic branches, the asymptotes to which are the projections of the tangents at the points where the line cuts the oval.

151. Let us now, in the first place, suppose that the line projected to infinity meets the infinite branch once, and does not meet the curve again, the oval will still remain a closed figure; but the branches of the infinite part, instead of spreading out in-

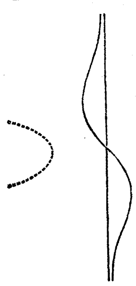
definitely, will approach to contact with a finite asymptote, and will assume *one or other* of the forms represented in the figure. The left-hand side represents the ordinary figure of the infinite branch. It obviously must have three points of inflexion; for the curve being convex towards the asymptote at positive infinity (since every curve is convex towards its tangent on both sides of the point of contact); it must change this convexity into concavity,



in order to cut the asymptote once again; having cut it, it must bend again, else it would continually recede from the asymptote, and it must bend once more in order to become convex towards the asymptote at negative infinity.

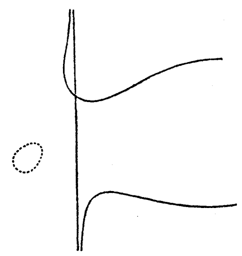
The right-hand figure represents the case when the curve has a point of inflexion at infinity. An ordinary asymptote to a curve (as the reader has seen in the case of the common hyperbola) has a positive and negative infinite branch at *opposite* sides of it; if, however, the asymptote touch at a point of inflexion, the two infinite branches lie *on the same* side of the line. It is obvious from the figure, that the curve has in this case but two finite points of inflexion. In the other cases I shall not think it necessary to give a separate figure for the case where there is one or more points of inflexion at infinity, but the reader may, if he pleases, substitute in any of the figures for an infinite branch cut by its asymptote, one lying altogether at the same side of its asymptote. I may add, that if the reader, in proving any theorem with regard to cubics, should wish to assist his imagination by drawing a figure, he will generally find the figure of this Article the most simple and convenient.

152. Next, let the line projected to infinity touch the curve. Two cases are to be distinguished according as this line touches the oval or the infinite part of the curve. In the first case, the infinite branch assumes one of the figures described in the last Article, while the oval takes a parabolic form.

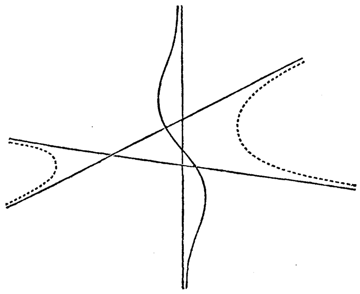


In the second case, the oval remaining as in the last Article, it is

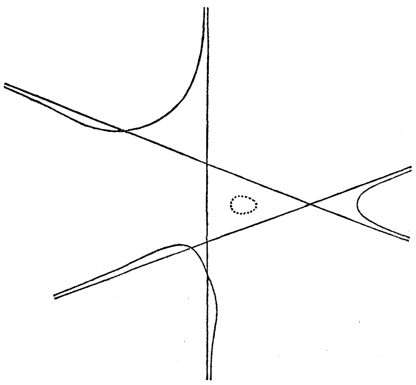
the infinite part which spreads into the parabolic form.



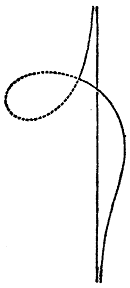
153. Lastly, let the line projected to infinity cut the curve in three real points. Here again there are two principal divisions of the figures to which this may give rise, according as two of these points are on the oval or not. If two of the points be on the oval, the infinite branch remains as before, and the oval assumes a form somewhat resembling the common hyperbola.



If the line do not meet the oval, in the projection there will be still an oval (if there had been one in the original figure) situated within the triangle formed by the asymptotes. Variations may exist in this figure; for example, the three asymptotes might intersect in a point, or two of the asymptotes might intersect on the curve; but for all practical purposes the division here made seems sufficient.



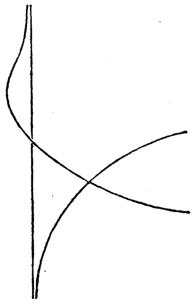
154. The preceding figures serve sufficiently to give an idea of the different forms which cubics of the sixth class, or cubics having a conjugate point, may possess. We shall next examine the different possible projections of the first figure, p. 30, which represents a cubic having a double point with real tangents. First, if the line projected to infinity meet the curve in only one

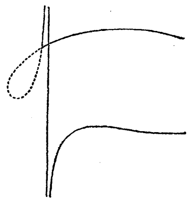


point, the only important change made in the figure is, that the branches, instead of spreading out indefinitely, approach to contact with a finite asymptote.

The same figure may serve to represent the projection of a cubic having a cusp, when the line projected to infinity meets the curve in only one point; if the reader imagine the loop removed, and the double point replaced by a cusp.

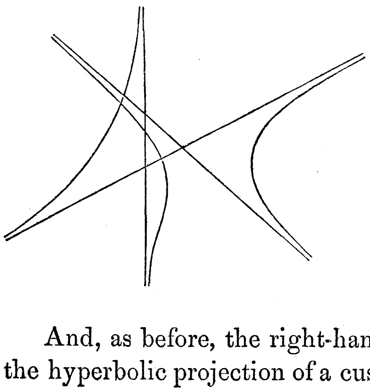
Secondly, if the line projected to infinity touch the curve. Two cases arise according as the line touches the looped part or the spreading part of the curve.

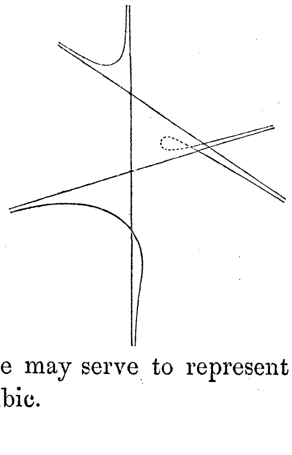




If in the right-hand figure the loop be supposed removed, and the double point replaced by a cusp, we have the case when a cusped cubic is projected into a parabolic form.

Thirdly, if the line projected to infinity cut the curve in three real and distinct points, we have again two distinct figures, according as the line cuts the looped part or not.

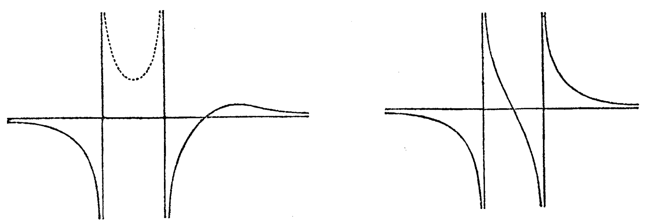




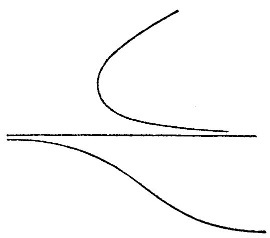
And, as before, the right-hand figure may serve to represent the hyperbolic projection of a cusped cubic.

Fourthly, if the double point be projected to infinity, the two tangents at the double point become parallel asymptotes.

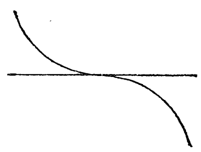
And again, we have two figures, according as the projected line meets the curve again in the loop or in the spreading part. In the latter case the point of inflexion is between; in the former, outside the two parallel asymptotes.



The left-hand figure may also represent the case when a cusp is projected to infinity, if the reader imagine the two parallel asymptotes to be united, and consequently the branch between them suppressed. The asymptote touching at a cusp has two infinite branches on opposite sides, but at the same end of it.



Fifthly, if not only the double point, but also the tangent at it, be projected to infinity, we obtain a curve called *the trident*, whose figure is here given.



If the tangent at a cusp be projected to infinity, we obtain the *cubical parabola*, a curve of the form here represented. In all the figures of this Article the reader will observe there is but one real point of inflexion.

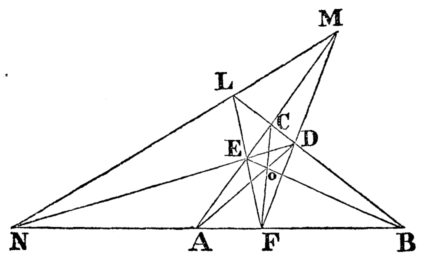
SECT. IV.—POLES AND POLARS.

155. It is easy to apply to curves of the third degree the theorems about poles and polars which have been proved already (p. 51, &c.). Every point has a polar line with respect to the curve; and the locus of the poles of all the right lines which can be drawn through a given point, is the polar conic of the given

point. Every right line has four poles. If, however, the curve be of the fourth class, it will have but three, since all polar conics must pass through the double point: if of the third class, every right line has but two poles. We shall first illustrate these poles and polars by the case where the curve of the third degree reduces itself to three right lines, $\alpha\beta\gamma = 0$.

The polar line of any point $\alpha'\beta'\gamma'$, with regard to this equation, is (p. 58)

$$\beta'\gamma'\alpha + \gamma'\alpha'\beta + \alpha'\beta'\gamma = 0, \text{ or } \frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0.$$



It will be seen that this is equivalent to the following geometrical construction: Join the point O to the vertices of the triangle, and then the polar is the line LMN on the figure. For AD, BE, CF are

$$\frac{\alpha}{\alpha'} = \frac{\beta}{\beta'} = \frac{\gamma}{\gamma'},$$

and EF, FD, DE are

$$\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} - \frac{\alpha}{\alpha'}, \quad \frac{\gamma}{\gamma'} + \frac{\alpha}{\alpha'} - \frac{\beta}{\beta'}, \text{ and } \frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'};$$

LMN is therefore

$$\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0.$$

The equation of the polar conic is

$$\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} + \frac{\gamma'}{\gamma} = 0,$$

a conic passing through the vertices of the given triangle. The tangent at any vertex is (*Conics*, p. 247) $\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} = 0$, and is therefore constructed by joining the vertex $\alpha\beta$ to the point where the polar line meets the opposite side γ.

156. Knowing how to construct the polar of a point with regard to a triangle, we can readily construct it with regard to any

curve of the third degree. For since two points determine the polar line, if through the given point O we draw a pair of lines cutting the curve in ABC, A′B′C′, the polar of O is the same with regard to all cubics which pass through these six points (Art. 60), and is therefore the same as with regard to the triangle formed by AA′, BB′, CC′. Thus we can, by the ruler alone, construct the polar of a given point with regard to a given cubic. As a particular case, it appears that the polar of a given point is the same as with regard to the triangle formed by the tangents at the points where any line through O meets the curve.

157. The problem to draw the tangents from a given point to a given cubic, reduces itself to constructing the polar conic of the given point with regard to the cubic.

Now, if through O we draw three right lines meeting the curve in $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, the polar conic of O is the same with regard to all cubics passing through these nine points, since we are given the two points where the polar conic meets each of the three radii.

If, therefore, we draw any transversal L cutting the curve in $A_1A_2A_3$, and draw a conic S through the other six points, the problem is reduced to finding the polar conic of O with regard to the system made up of the conic S and the right line L. But

$$d(\mathrm{SL}) = \mathrm{L}d\mathrm{S} + \mathrm{SL}'.$$

Hence the required polar conic passes through the intersections of L and S, and also through the points where S is met by the polar of O with regard to S. This suffices to determine the conic, since by the last Article we can construct the polar line, which is also the polar of O with regard to the required conic (Art. 58).

158. If from two consecutive points of the curve we draw the two sets of tangents, OA, OB, OC, OD; PA, PB, PC, PD; any tangent, OA, intersects the consecutive tangent PA in its point of contact A; but it has been proved (Art. 70) that the four points of contact, ABCD, lie on the polar conic of O, which also touches the curve at the point O (Art. 65); hence the six points, OPABCD, lie on the same conic, and therefore the anharmonic ratio of the pencil {O.ABCD} is the same as that of the pencil

{P.ABCD}. Since, then, this ratio remains the same when we pass from one point of the curve to the consecutive one, we learn that *the anharmonic ratio is constant of the pencil formed by the four tangents which may be drawn from any point of the curve.**

The ratio of this pencil may serve as a distinguishing numerical characteristic of the curve; and since the value of an anharmonic function is unaltered by projection, no two cubics can be projected into each other unless their characteristics be the same.

It follows also from this theorem, that if O, P be *any* two points of the curve, through these points can be drawn a conic passing through the four points, where each of the tangents from the first point meets the corresponding tangent from the second.

The anharmonic ratio of four points, *abcd*, is unaltered by writing them in the order *badc* or *cdab* or *dcba*; hence, by taking the legs of the second pencil successively in each of these four orders, we see that the sixteen points of intersection of the first set of tangents with the second, lie on four conics passing through the points OP.

159. As the polar conic of a point, with regard to a cubic, has been defined as the locus of the poles of all right lines passing through that point, so we may speak of the polar conic of a right line, meaning thereby *the envelope of the polars of all the points on that right line.*

Let the equation of the right line be $ax + by + cz = 0$, and denoting the second differential coefficients by ABCDEF, the equation of the polar line of any point, $x'y'z'$, is

$$Ax'^2 + 2Bx'y' + Cy'^2 + 2Dx'z' + 2Ey'z' + Fz'^2 = 0, \text{ or } V = 0;$$

and the envelope of this, subject to the condition $ax' + by' + cz' = 0$, is (*Conics*, p. 328)

$$(E^2 - CF)a^2 + (D^2 - AF)b^2 + (B^2 - AC)c^2$$
$$+ 2(BF - DE)ab + 2(CD - BE)ac + 2(AE - BD)bc = 0.$$

This equation may be used to find the condition that the line $ax + by + cz$ should touch the given cubic. For since the polar of

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* This theorem was suggested to me by Dr. Hart's theorem, that the four foci of a circular cubic lie on a circle. The proof here given, which is obvious enough, presented itself to both of us at the same time.

any point on the curve is the tangent at that point, if $ax + by + cz$ touch the cubic, it must also touch its polar conic. If, then, by substituting for the second differential coefficients their values in xyz, we give to the equation just written the form

$$\mathrm{a}x^2 + 2\mathrm{b}xy + \mathrm{c}y^2 + 2\mathrm{d}xz + 2\mathrm{e}yz + \mathrm{f}z^2 = 0,$$

the condition required will be (*Conics*, p. 328)

$$(\mathrm{e}^2 - \mathrm{cf})a^2 + (\mathrm{d}^2 - \mathrm{af})b^2 + (\mathrm{b}^2 - \mathrm{ac})c^2 + 2(\mathrm{bf} - \mathrm{de})ab + 2(\mathrm{cd} - \mathrm{bc})ac + 2(\mathrm{ae} - \mathrm{bd})bc = 0.$$

This is the geometrical translation of Hesse's method of finding the reciprocal of a curve of the third order, alluded to, note, p. 101.

160. It is worth remarking, that the polar conic of a line is also the locus of the pole of that line with regard to the polar conics of its several points. For the latter problem is solved by eliminating $x'y'z'$ between the equations

$$\frac{\dfrac{dV}{dx'}}{a} = \frac{\dfrac{dV}{dy'}}{b} = \frac{\dfrac{dV}{dz'}}{c};$$

and it will be seen that it is the very same system of equations by the help of which the proposed envelope would be found.

The most interesting application of this is where it is required to find the polar conic of the line at infinity; or in other words, the envelope of all the diameters. Making $a = 0$, $b = 0$, in the preceding, we get for its equation

$$\left(\frac{d^2U}{dxdy}\right)^2 = \left(\frac{d^2U}{dx^2}\right)\left(\frac{d^2U}{dy^2}\right).$$

It follows also from the remark just made, that *the envelope of all the diameters of a curve of the third degree is also the locus of the centres of all the diametral conics.*

161. Since the polar line of any point on a given line is the same as if taken with regard to the three tangents at the points where that line meets the curve, it appears that the polar conic of a line is the same as if taken with regard to these three tangents. Moreover, since these three tangents are the polar lines of their points of contact which lie on the given line, it appears that the sought envelope must touch these three lines.

The same thing will appear by investigating directly the polar conic of a line with regard to the triangle $\alpha\beta\gamma$.

But this is to find the envelope of

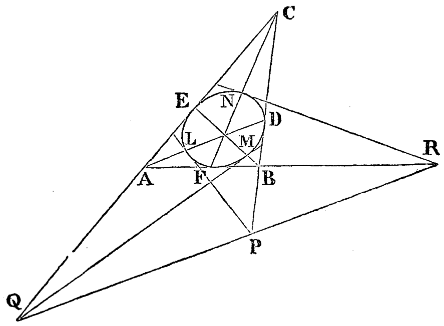
$$\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0,$$

subject to the condition

$$L\alpha' + M\beta' + N\gamma' = 0,$$

and this (*Conics*, p. 241) is

$$(L\alpha)^{\frac{1}{2}} + (M\beta)^{\frac{1}{2}} + (N\gamma)^{\frac{1}{2}} = 0.$$



It follows, then, that if the given line be PQR, the conic is that touching the sides of the triangle at the points DEF, where each side is cut harmonically in the points CDBP, &c.

162. It follows, as a particular case of the last, that the polar conic of the line at infinity is the conic touching the asymptotes, at the three middle points of the sides of the triangle formed by them.

Now it follows at once, from the definitions, that the two tangents which can be drawn from any point to the polar conic of a right line are the polars of the two points where the polar conic of the point meets the right line. Hence the two tangents will be real when the polar conic meets the right line in real points. Applying this to the case when the right line is at infinity, we learn that *the polar conic of any point is an ellipse when that point is situated within the ellipse touching the triangle formed by the asymptotes at the middle points of its sides; the polar conic of any point outside that ellipse is a hyperbola, and of any point on that ellipse is a parabola.*

The polar conic of a double point is the pair of tangents at the double point; hence, as a particular case of the last, we have M. Plücker's theorem, *If a cubic have a node it must lie* outside; *if a conjugate point,* inside; *if a cusp,* on *the ellipse, touching at their middle points the sides of the triangle formed by the asymptotes.*

Hence, also, when a cubic has a cusp, the polar conic of every line must pass through the cusp.

163. We have seen that the curve H is the locus of points whose polar conics break up into two right lines, and tangents from which have their points of contact lying on systems of two right lines.

If the polar conic of any point, E, *be two right lines passing through* F, *then the polar conic of* F *is two right lines passing through* E.

For if $x'y'z'$ be the co-ordinates of E, the equation of its polar conic may be written in either of the equivalent forms,

$$(1.)\quad V = x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} = 0,$$

or
$$(2.)\ x^2\left(\frac{d^2U}{dx^2}\right)_1 + y^2\left(\frac{d^2U}{dy^2}\right)_1 + z^2\left(\frac{d^2U}{dz^2}\right)_1 + 2xy\left(\frac{d^2U}{dxdy}\right)_1 + 2yz\left(\frac{d^2U}{dydz}\right)_1 + 2zx\left(\frac{d^2U}{dxdz}\right)_1 = 0;$$

and if this break up into two right lines, the co-ordinates of F must fulfil the relation $\frac{dV}{dx} = 0$, $\frac{dV}{dy} = 0$, $\frac{dV}{dz} = 0$; but by performing these differentiations on the equation, first in the shape (1), and secondly in the shape (2), we see that the relation between the points is reciprocal, for that we get equations of precisely the same form to determine xyz from $x'y'z'$, as *vice versâ.*

In general, the system of equations

$$A'x + B'y + D'z = 0,\quad B'x + C'y + E'z = 0,\quad D'x + E'y + F'z = 0,$$

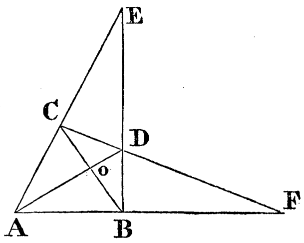
(where $A'B'$, &c. are the second differential coefficients) express either the relations which exist when the polar conic of $x'y'z'$ break up into two right lines, or when the first polar of xyz has a double point. Eliminate xyz, and we get $H = 0$, which is either the locus of all possible double points on first polars, or else of points whose polar conics have a double point. Eliminate $x'y'z'$, and we should get the locus of points whose first polars have double points, or of all possible double points on polar conics. Now it is only in the case of curves of the third degree that these two results of elimination are the same.

164. In curves of the third degree, *the tangent at any point* E *of the curve* H *is the polar of the corresponding point* F, *with regard to the given curve* U.

For, take two consecutive positions of the point E, and the poles of the line joining them are the intersections of the two corresponding polar conics; but it is plain that two of these intersections coincide with F. (Hence H may also be defined as the envelope of the polar lines of all points whose polar conics break up into two right lines.) The other two intersections are the points where each of the lines which make up the polar conic of E is met by the consecutive line, so that if we formed the envelope of these lines, each point of the envelope would be the pole of a tangent to H.

The two tangents to H *at corresponding points,* E, F, *meet on the curve* H.

Let the polar conic of E be FC, FA, and let the polar conic of F be EA, EB; then ABCD are the four poles of the line EF, and the polar conic of every point of the line EF passes through these four points. If, therefore, this polar conic break up into two right lines, the two right lines must be AD, BC; and we see that O is a point on the curve H, and



corresponds to the point in which EF meets that curve again. But the tangent to H at E is the polar of F with regard to the curve U, which must be also its polar (Art. 58) with regard to the polar conic of F (EA, EB); therefore, by the harmonic properties of a quadrilateral this tangent is the line EO; similarly the tangent at F is the line FO. Q. E. D.

165. It will be proved in the last section of this Chapter that the problem admits of three solutions, "Given the curve H to find the curve U, of which H is the Hessian determinant." And if at any point E of H we draw the tangent EO to meet H again in O, and from O draw the three other tangents to H, it appears from the last Article that the points of contact are the points corresponding to E, according as H is supposed to be the Hessian of

one or other of these three curves. When we have once selected which point of contact we shall take as corresponding to any point E of the curve, the theorem of Art. 158 removes all ambiguity in the case of any other point G of the curve; for by that theorem, if from two points of the curve O, P there be drawn sets of four tangents, and if one of the first set EO be taken as corresponding to one of the other set GP, the order in which the other tangents correspond to each other is quite determinate.

166. *When several cubics pass through the same nine points, the polars of any point, with regard to all the curves of the system, pass through a fixed point.*

For the polar of any point $x'y'z'$, with regard to $S + kS' = 0$, is

$$\left\{x\left(\frac{dS}{dx}\right)_{,} + y\left(\frac{dS}{dy}\right)_{,} + z\left(\frac{dS}{dz}\right)_{,}\right\} + k\left\{x\left(\frac{dS'}{dx}\right)_{,} + y\left(\frac{dS'}{dy}\right)_{,} + z\left(\frac{dS'}{dz}\right)_{,}\right\} = 0,$$

involving a variable quantity only in the first degree.

In like manner, all the polar conics of a fixed point, with regard to cubics of the system, pass through four fixed points.

It is obviously true in general that the polar curves of a fixed point, with regard to curves of any degree, included in a system $S + kS' = 0$, pass through certain fixed points.

167. *The tangent at any point of a cubic meets the curve again, where it meets the polar of the given point, with regard to its Hessian determinant.*

We saw (Art. 164) that the tangent to H, at a point E, meets H again, where it meets the polar of E with regard to U; and we see, from the last Article, that through the same point will pass the polar of E with regard to any curve of the system $U + kH$. But (Art. 165) H may be taken for any curve of the third degree, and (Art. 87) its Hessian H (HU) is of the form $U + kH$.

This leads us again to the theorem obtained already (Art. 94), that the co-ordinates of the point where the tangent meets the cubic again, are found by combining the equations

$$x\frac{dU}{dx} + y\frac{dU}{dy} + z\frac{dU}{dz} = 0, \quad x\frac{dH}{dx} + y\frac{dH}{dy} + z\frac{dH}{dz} = 0.$$

SECT. V.—DOUBLE POINTS.

168. We have proved (Art. 74) that the condition that the equation of the third degree should represent a cubic having a double point, is obtained by eliminating the variables between $\frac{dU}{dx}=0$, $\frac{dU}{dy}=0$, $\frac{dU}{dz}=0$, and is of the twelfth degree in the coefficients. In the last section of this Chapter we shall speak of the actual formation of this condition, but there are some considerations respecting double points, which we shall here mention. From the mere degree of the condition it follows, that through eight given points can be described twelve cubics having double points. For, if we apply this condition to the equation $S+kS'=0$, we get an equation of the twelfth degree to determine k (see *Conics*, p. 216). The same thing may be otherwise seen thus: the three following conditions must be fulfilled for a double point,

$$\frac{dS}{dx}+k\frac{dS'}{dx}=0,\quad \frac{dS}{dy}+k\frac{dS'}{dy}=0,\quad \frac{dS}{dz}+k\frac{dS'}{dz}=0.$$

The double point must then lie on each of the curves found by eliminating k between any two of these equations, viz.:

$$\frac{dS}{dx}\cdot\frac{dS'}{dy}=\frac{dS}{dy}\cdot\frac{dS'}{dx},\quad \frac{dS}{dy}\cdot\frac{dS'}{dz}=\frac{dS}{dz}\cdot\frac{dS'}{dy},\quad \frac{dS}{dz}\cdot\frac{dS'}{dx}=\frac{dS}{dx}\cdot\frac{dS'}{dz}.$$

These are curves of the fourth degree, any pair of which will intersect in sixteen points; but the first pair have common the four points $\frac{dS}{dy}$, $\frac{dS'}{dy}$, which do not lie on the third. Hence there are but twelve points common to all three.*

169. *To find the points whose polars are the same with regard to all cubics passing through eight given points.*

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* In general, when from three equations $\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}$ it is inferred that the three curves $AB'-A'B=0$, $BC'-CB'=0$, $CA'-AC'=0$, have common points, it must be observed that *all* their points of intersection are not common; for any values which made both numerator and denominator of any of these fractions to vanish, will satisfy two of the resulting equations, but not the third.

It follows from Art. 166, that such points are determined by the conditions

$$\frac{\frac{dS}{dx}}{\frac{dS'}{dx}} = \frac{\frac{dS}{dy}}{\frac{dS'}{dy}} = \frac{\frac{dS}{dz}}{\frac{dS'}{dz}};$$

for then the two equations

$$x\left(\frac{dS}{dx}\right)_{,} + y\left(\frac{dS}{dy}\right)_{,} + z\left(\frac{dS}{dz}\right)_{,} = 0,\ x\left(\frac{dS'}{dx}\right)_{,} + y\left(\frac{dS'}{dy}\right)_{,} + z\left(\frac{dS'}{dz}\right)_{,} = 0$$

will represent the same right line.

But these conditions are the very same which we found in the last Article to determine the twelve points capable of being double points on cubics belonging to the system $S + kS' = 0$. The same demonstration proves, for curves of any degree, that the polar of any point capable of being a double point on one of the curves of the system $S + kS'$, will be the same with regard to all curves of the system. The reader is already familiar with this theorem in the case of conic sections.

170. We add another investigation, which will lead to the same result as Art. 168. *To find the locus of the double points of all cubics which pass through seven fixed points.*

Let A, B, C be three cubics passing through the seven points, then any other will be

$$A + kB + lC = 0,$$

and the double point must satisfy

$$\frac{dA}{dx} + k\frac{dB}{dx} + l\frac{dC}{dx} = 0,\quad \frac{dA}{dy} + k\frac{dB}{dy} + l\frac{dC}{dy} = 0,$$

$$\frac{dA}{dz} + k\frac{dB}{dz} + l\frac{dC}{dz} = 0;$$

whence eliminating k, l, we have the equation of the locus

$$\frac{dA}{dx}\left(\frac{dB}{dy}\frac{dC}{dz} - \frac{dB}{dz}\frac{dC}{dy}\right) + \frac{dA}{dy}\left(\frac{dB}{dz}\frac{dC}{dx} - \frac{dB}{dx}\frac{dC}{dz}\right)$$
$$+ \frac{dA}{dz}\left(\frac{dB}{dx}\frac{dC}{dy} - \frac{dB}{dy}\frac{dC}{dx}\right) = 0,$$

an equation of the sixth degree. It is plain that each of the seven given points is on the locus, since a cubic can be described

(Art. 36), having one of them for a double point, and passing through the six others. But, moreover, these seven points are also double points on the locus; for if the equations A, B, C be written at full length; and if the coefficients of x and y be H, K, the coefficient of x in the new equation will be

$$H(H'K'' - H''K') + H'(H''K - HK'') + H''(HK' - H'K),$$

which vanishes identically; and so likewise does the coefficient of y. It will readily be seen that the tangents to the locus at the double point are the same as the tangents to the cubic which has that point for a double point.

Now, then, let it be required to find the number of cubics having double points, which can be drawn through eight given points. First, suppose the locus constructed of double points on cubics passing through all but the eighth, then through all but the seventh; these two loci being of the sixth degree, must intersect in thirty-six points; but they have common six points, which being double points on both, count for twenty-four (Note, p. 31); there remain, therefore, but twelve other points common to both curves.

171. Of these twelve points, the position of some may in particular cases be at once perceived. Thus in the case

$$ABC + kDEF = 0,$$

since one cubic of the system is ABC, having the three double points AB, BC, CA, and another cubic of the system is DEF, having three double points in like manner, there are but six other double points.

We shall give a more particular examination to the equation

$$ABC + kD^2F = 0,$$

in which case we shall presently show that there are but *three* other double points. And as to their position, it is plain, in the first place, since, by Art. 169, the polar of any of the required double points is the same with regard to ABC and to D^2F, and since the polar line of any point with regard to D^2F passes through the point DF, that the required points must all lie on the polar conic of the point DF, with regard to the triangle ABC.

We can determine other loci, on which the required double

points must lie, if we construct the locus of the points of contact of tangents to curves of the system drawn from each of the points AF, BF, CF. For when a curve has a double point, every line through that point must be considered as there touching the curve (Art. 30). These three loci, therefore, the method of finding which we give in the next Article, must all pass through the points required.

172. If through the point CF we draw any line $C = \lambda F$, this must meet the cubic $ABC = kD^2F$, where it meets the conic $\lambda AB = kD^2$. Hence, given any point, m, on the cubic, we can construct an infinity of others. For if we suppose a conic drawn through m, and touching the lines A, B, where they meet D, we can, by linear constructions (*Conics*, p. 284), find the point where the line joining m to CF meets this conic again. And it appears from what we have said that this must be also a point on the cubic. So in like manner we can construct the points where the lines joining m to AF and to BF meet the cubic again. And by repeating the construction with these new points, we can find an infinity of others. We see also that if the line $C = \lambda F$ touch the cubic, it must also touch the conic $\lambda AB = kD^2$; and the problem of the last Article is reduced to "finding the locus of the point of contact of tangents from a fixed point, CF, to a system of conics touching the lines A, B at the points AD, BD." The nature of this locus is at once seen geometrically by projecting the system of conics into concentric circles, or else, algebraically, as follows: The tangent to a conic $AB = k^2D^2$ is (*Conics*, p. 228),

$$\mu^2 A - 2\mu k D + B = 0,$$

and at the point of contact we have $\mu^2 = \frac{B}{A}$, $\mu k = \frac{B}{D}$. Now if we substitute in the equation of the tangent, the co-ordinates of the fixed point, and put in these values of μ^2, μk, which are satisfied for the point of contact, we obtain for the locus of this point

$$\frac{A'''}{A} - 2\frac{D'''}{D} + \frac{B'''}{B} = 0,$$

and we see that the required locus is a conic circumscribing the triangle ABD, and passing also through the fixed point.

The locus of points of contact of tangents from BF is, in like manner,

$$\frac{A''}{A} - 2\frac{D''}{D} + \frac{C''}{C} = 0,$$

and has common with the preceding locus, the point AD. But this is not a point on the locus of points of contact of tangents from AF,

$$\frac{B'}{B} - 2\frac{D'}{D} + \frac{C'}{C} = 0.$$

In fact, AD is on the cubic $ABC = kD^2F$, a point which has always A for its tangent, and only comes in as a point on the locus, because it is a double point for the case of $k = \infty$; but we are at present seeking the double points which the system may possess independently of the cases $k = 0$, $k = \infty$. The three loci, then, can have only three points of intersection common to them all, and there are, consequently, but three double points on the system.

173. The reader may wish to see it verified algebraically that the three loci have common points of intersection.

Let $$D = lA + mB + nC, \quad F = l'A + m'B + n'C.$$

Then the co-ordinates of AF may be taken

$$A' = 0, \quad B' = n', \quad C' = -m', \quad D' = mn' - nm'.$$

Substituting these values, the equation

$$\frac{B'}{B} - 2\frac{D'}{D} + \frac{C'}{C} = 0$$

becomes

$$\frac{n'}{B} - \frac{m'}{C} = 2\frac{mn' - nm'}{D}.$$

In like manner the other two loci are

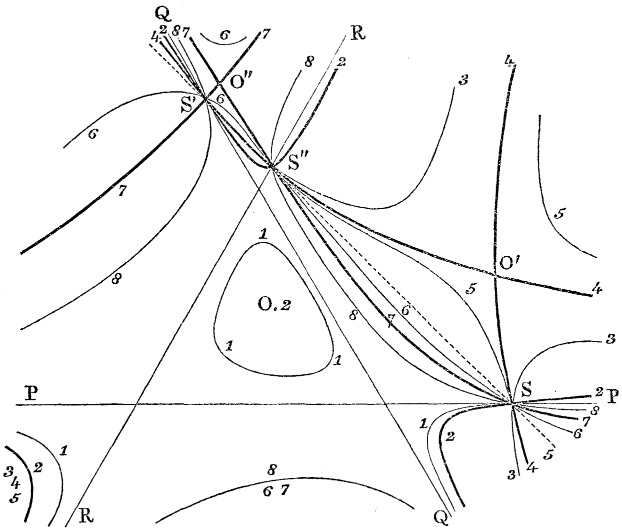
$$\frac{l'}{C} - \frac{n'}{A} = 2\frac{nl' - ln'}{D},$$

$$\frac{m'}{A} - \frac{l'}{B} = 2\frac{lm' - ml'}{D},$$

equations which, when multiplied respectively by l', m', n', and added together, identically vanish.

174. We have given this discussion, on account of its importance (when the line D is supposed at infinity), in M. Plücker's classification of curves of the third degree. His method is to take

the equation referred to the asymptotes and the line joining the points where they meet the curve again ($ABC = k^2F$), and to trace the different forms which the curve assumes as we give all possible values to the constant k. By repeating this process with every variety of position which the line F can have with regard to the triangle ABC, he obtains all possible forms of cubics having three real asymptotes. The cases of coincident or imaginary asymptotes are discussed in like manner. The reader can then readily understand the importance of attending to the values of k which give a double point on the curve, since these are cases of transition from one form of the curve to another. To make this more obvious, I have copied one of M. Plücker's figures. In the figure, S, S′, S″ are the points where the asymptotes meet the curve, which are given. O, O′, O″ are the possible double points. Fig. 1



represents a curve with an oval, such as is represented in the second figure of Art. 153. As the constant k is altered, the oval shrinks into the conjugate point O. As the constant is further altered, the figure (3) remains much the same, but there is no oval, and the branches recede further from their asymptotes. In figure 4 the branches cross to the other asymptotes, intersecting at the double point O′. In figure 5 there is one continuous

infinite branch touching the asymptote QQ, accompanied by the hyperbolic projection of an oval, as in the first figure, Art. 153. Figure 6 is of the same nature, only that PP is the asymptote touched by the continuous infinite branch. In figure 7 O″ becomes a double point, and figure 8 is of the same species as figure 3, only that the points of inflexion belong to branches which touch different asymptotes. These are the first eight of M. Plücker's 219 species, and the reader will see, on referring to Sect. III. of this Chapter, that we regard 3 and 8, 4 and 7, 5 and 6, as not essentially distinct from each other. We do not think it necessary to discuss his other species in like manner, although we have thought that the reader might be interested by a single illustration of the manner in which the figure of a curve is changed by the alteration of a constant in its equation.

175. "If any of the points O, O′, O″ be joined to any of the points AF, BF, CF, the point O will be the middle point of the chord intercepted between the two points where this line meets again any cubic of the system." This readily appears from applying the method of Art. 172 to the case where D is at infinity, and from the property that the intercepts on any chord of a hyperbola between the curve and the asymptotes are equal. M. Plücker has hence named the points O, O′, O″ *middle points* of any of the cubics having A, B, C for asymptotes, and passing through AF, BF, CF.

176. M. Plücker has also shown that *any of the points* O, O′, O″ *is the pole of the line joining the other two, with regard to the polar conic of the line* D *with regard to the triangle* ABC. And it may be observed that this polar conic is the same with regard to any cubic of the system. As I have not been able to perceive the geometrical reason for the truth of M. Plücker's theorem, the following algebraical proof must suffice. Let us suppose one of the points given $A = B = C$ (constants being supposed implicitly contained in the equations of the lines), and it is required to find the other two. Since this point is to be a point of contact given (Art. 172) by equations of the form $\mu^2 = \frac{B}{A}$, $\mu k = \frac{B}{D}$, we must

have (if $D = lA + mB + nC$) $\mu = 1$, $k = \dfrac{1}{l+m+n}$, and the corresponding tangent ($\mu^2A - 2\mu kD + B$) becomes

$$(l+m+n)(A+B) = 2(lA+mB+nC).$$

The co-ordinates of the point CF, where this line meets C, must be proportional to the following,

$C''' = 0$, $B''' = -(m+n-l)$, $A''' = l+n-m$, $D''' = (l-m)(l+m+n)$,

and the locus of points of contact of tangents through CF is (Art. 172)

$$\frac{l+n-m}{A} - \frac{m+n-l}{B} = 2\frac{(l-m)(l+m+n)}{lA+mB+nC},$$

which may be also written

$$\{(m+n-l)A-(l+n-m)B\}\{lA-(l+n)B+nC\} + (l+m+n)(l+n-m)(A-B)B = 0.$$

In like manner the locus of points of contact of tangents through AF is

$$\{(l+m-n)C-(l+n-m)B\}\{lA-(l+n)B+nC\} + (l+m+n)(l+n-m)(C-B)B = 0.$$

Multiply the first equation by l, and the second by n, and add; the sum becomes divisible by $lA-(l+n)B+nC$, and we have for the line joining the other two possible double points,

$$(m+n-l)lA + (n+l-m)mB + (l+m-n)nC = 0\,;$$

but this is the polar of the point $A = B = C$, with regard to the polar conic of $lA+mB+nC$, whose equation is (Art. 161)

$$(lA)^{\frac{1}{2}} + (mB)^{\frac{1}{2}} + (nC)^{\frac{1}{2}} = 0,$$

or $$l^2A^2 + m^2B^2 + n^2C^2 - 2lmAB - 2mnBC - 2nlAC = 0.$$

Two of the double points coincide when either is on the polar conic, and in this case we have seen (Art. 162) that the double point will be a cusp.

SECT. VI.—CUBICS OF THE THIRD CLASS.

177. Cubics which have a cusp admit of some simple methods of treatment, analogous to those applied to conic sections (*Conics*,

p. 227), and therefore may conveniently receive a separate discussion. Their equation can always be reduced to the form

$$L^2M = R^3,$$

where L is the tangent at the cusp, and M that at the point of inflexion. Any point on the curve is the intersection of the two lines $\mu L = R$, $\mu^2 R = M$, where μ is a variable parameter.

[We may remark that these equations, considered as belonging to tangential co-ordinates, give the theorem: "Any tangent, AB, cuts the sides of the triangle ICT, formed by the tangents at the cusp and at the point of inflexion, so that $\frac{IA^2}{AT^2} = k\frac{TB}{BC}$; and when the line at infinity is a tangent (Art. 7) $k = 1$;" a theorem somewhat resembling the well-known property of the parabola (*Conics*, p. 274)].

The line joining any two points of the curve will have for its equation

$$\mu\mu'(\mu + \mu')\,L - (\mu^2 + \mu\mu' + \mu'^2)\,R + M = 0.$$

Let the points $\mu\mu'$ coincide, and we see that the equation of the tangent at any point is

$$2\mu^3 L - 3\mu^2 R + M = 0.$$

178. We can now easily express the relation between three points of the curve which are on a right line. It is simply $\mu + \mu' + \mu'' = 0$. For let the equation of the right line be $M = lL + rR$, substituting this in the equation of the curve, we get, to determine the μ's of the points where the line meets the curve, $l + r\mu = \mu^3$, an equation in μ wanting its second term, and which, therefore, has the sum of its roots $= 0$. Hence, in this notation, the point where the tangent at μ meets the curve again is -2μ, and the point of contact of the tangent from μ is $-\frac{1}{2}\mu$.

This theorem may be generalized, and the relation which exists between the $3p$ points, where a curve of the p^{th} degree meets the cubic, is $\Sigma\mu = 0$. For if we substitute in the equation of the curve of the p^{th} degree, for M, $\frac{R^3}{L^2}$, and then put $\frac{R}{L} = \mu$, it will be found that the equation in μ will want the second term. The highest power in μ will arise from the term M^p, and will be μ^{3p}; the next highest will arise from the term $M^{p-1}R$, and will be μ^{3p-2}.

We see thus (as we found already, Art. 26) that the $3p$ points, where the curve of the p^{th} degree meets the cubic, are such, that all but one being given, that one may be found.

179. The facility which this mode of expressing points on a cubic of the third class, gives to the management of questions concerning them, renders them nearly as easy to be treated as conic sections. Many theorems, too, are thus suggested, which, though only thus *proved* for curves of the third class, if they do not involve any mention of the cusp or of the number of tangents which can be drawn to the curve from any point, we may see will be true for all curves of the third degree.

Ex. 1. *A chord is drawn through a point on a cubic, and tangents where it meets the curve; find the locus of their intersection.* The tangents will be

$$2\alpha^3L - 3\alpha^2R + M = 0,$$
$$2\beta^3L - 3\beta^2R + M = 0,$$

where $\alpha + \beta + \gamma = 0$, and γ is known.

We eliminate α, β thus: first, subtract the equations, and we get

$$2(\gamma^2 - \alpha\beta)L + 3\gamma R = 0.$$

Multiply the first by β^2, the second by α^2, and subtract; and we have

$$2\alpha^2\beta^2L + \gamma M = 0,$$

whence eliminating $\alpha\beta$, we find

$$\gamma(2\gamma L + 3R)^2 + 2LM = 0,$$

the equation of a conic.

Ex. 2. *If a polygon of an even number of sides be inscribed in a cubic, and all the sides but one pass through fixed points on the curve, the last side will also pass through a fixed point on the curve.*

Let us denote the μ's of the vertices by $a_1a_2a_3$, &c., and of the fixed points by b_1b_2, &c. We take the case of the quadrilateral for simplicity, but the proof is just the same in general. We have then the equations

$$a_1 + b_1 + a_2 = 0, \quad (1)$$
$$a_2 + b_2 + a_3 = 0, \quad (2)$$
$$a_3 + b_3 + a_4 = 0, \quad (3)$$
$$(1) - (2) + (3) = a_1 + (b_1 + b_3 - b_2) + a_4 = 0;$$

or the last side passes through a fixed point, b_4, such that $b_1 + b_3 = b_2 + b_4$, or such that the lines joining b_1, b_3 and b_2, b_4 meet on the curve. It is not difficult to give an independent proof of this theorem for all curves of the third degree.

Ex. 3. *To inscribe in a cubic a polygon of an odd number of sides whose sides pass each through fixed points on the curve.*

Take, for simplicity, the case of the triangle, and we have

$$a_1 + b_1 + a_2 = 0,$$
$$a_2 + b_2 + a_3 = 0,$$
$$a_3 + b_3 + a_1 = 0,$$
$$(1) + (3) - (2) = 2a_1 + (b_1 + b_3 - b_2) = 0,$$

whence we have the following construction: a_1 is the point of contact of a tangent from $(b_1 + b_3 - b_2)$, which is constructed by joining b_1b_3, and joining the point where this line meets the curve again to b_2; the point where this latter line meets the curve again will be $(b_1 + b_3 - b_2)$. Or we might, by supposing two of the vertices of the polygon to coincide, have inferred from the last example that, if the sides of a polygon of an odd number of sides pass through fixed points on the curve, the tangent at any vertex passes through a fixed point on the curve. The problem, therefore, of this example admits, in the most general case, of four solutions.

Ex. 4. We have seen that the following relation subsists between the six points of intersection of a conic and a cubic:

$$(a + b + c + d) + e + f = 0.$$

Hence if four of them be given, the chord joining the remaining two passes through the fixed point $(a + b + c + d)$, which we have called (Art. 132) the opposite of the four given points. It is obviously constructed by joining ab, cd, and then joining the two points where these lines meet the curve again.

Ex. 5. *Given eight points on a cubic, to construct the ninth point where any other cubic through the points meets the curve again.*

We have $$(a + b + c + d) + (e + f + g + h) + i = 0,$$

whence it is plain we have only to construct the opposites of two sets of four, and produce the line joining these last to meet the curve again.

The relation $(a+b+c)+(d+e+f+g+h+i)=0$
shows at once that if three of the points be on a right line, the other six will be on a conic section, and *vice versâ*.

The reader will have no difficulty in seeing how, in like manner, given $3p-1$ points on a curve of the third degree, to find by the ruler alone the point where a curve of the p^{th} degree through these points would meet the curve again.

180. The solution of the last Article only enables us, "given eight of the points of intersection of two cubics, to find the ninth," when we have one cubic drawn through the eight points. For the solution of the general case I am indebted to Dr. Hart. It must be noticed, in the first place, that the equation

$$(a+b+c+d)+e+f+g+h+i=0$$

expresses that the *opposite* of four of the points lies on the conic passing through the other five. Now let the equation of the cubic be $AS=BT$, where S goes through *abcde*, T through *abcdf*; A is an indeterminate line through f, and B an indeterminate line through e; and the form of the equation shows that AB will be the point $a+b+c+d$. Then substituting in the equation successively the co-ordinates of g, h, and denoting the result of the substitution by S_1, S_2, &c., we have

$$\frac{S_1T_2}{S_2T_1}=\frac{B_1A_2}{A_1B_2},$$

the left-hand side of the equation is known, and therefore also the other. But the right-hand side of the equation expresses the anharmonic ratio of the segments of the line gh made by the lines AB. We are therefore given the anharmonic ratio of the pencil joining *efgh* to $a+b+c+d$, and therefore are given the conic through these points and i. The intersection of this conic with another similarly determined gives the point i. For the manner in which Dr. Hart performs this construction by the ruler alone, see Cambridge and Dublin Mathematical Journal, vol. vi. p. 181.

181. From the equation of the tangent $2\mu^3L-3\mu^2R+M=0$ it appears that if there be three points on the curve, such that the tangents at them meet in a point, we must have $\frac{1}{\mu}+\frac{1}{\mu'}+\frac{1}{\mu''}=0$;

for if we substitute in the equation of the tangent the co-ordinates of any point, the cubic equation we get for μ wants the coefficient of μ.

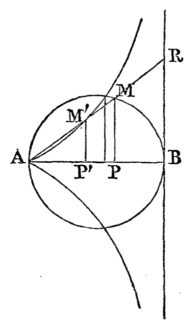
Again, we can prove that if there be $3p$ points whose tangents touch a curve of the p^{th} class, we must have the relation $\Sigma \frac{1}{\mu} = 0$. For (Art. 2) a curve of the n^{th} class is represented by an equation of the n^{th} degree in the coefficients of the equation of the tangent ($2\mu^3$, $3\mu^2$, and 1), and it is evident that such an equation must want the term μ.

182. It remains to mention a few of the more remarkable examples of cubics of the third class. We have already noticed the *semicubical parabola*, which is the evolute of the parabola of the second degree. In its equation, $py^2 = x^3$, the cusp is at the origin, and the point of inflexion at infinity. In the *cubical parabola*, on the other hand, $p^2y = x^3$, the point of inflexion is at the origin, and the cusp at infinity. In the cubical parabola the origin is a centre, and all the diameters of the curve coincide with the axis of y; for if we draw any line $y = mx + n$, the sum of the values of x is $= 0$.

The semicubical parabola has but one focus which coincides with that of the parabola of which it is the evolute (see Art. 127); the cubical parabola has two foci, whose co-ordinates are

$$27x'^2 = 27y'^2 = 2p^2.$$

To the cusped class also belongs the *cissoid of Diocles*, a curve imagined by that geometer for the solution of the problem of finding two mean proportionals. It may be defined as the locus of a point M′, where the radius vector to the circle AM is cut by an ordinate, such that AP′ = BP. We must have



$$AM' = RM, \text{ and therefore } \rho = AR - AM,$$

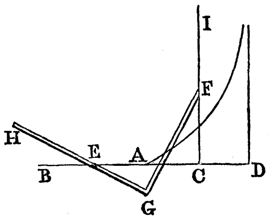
or $$\rho = 2r\sec\omega - 2r\cos\omega = 2r\tan\omega\sin\omega;$$

or, in rectangular co-ordinates,

$$x(x^2 + y^2) = 2ry^2, \text{ or } (2r - x)y^2 = x^3.$$

The origin is therefore a cusp, and $2r - x$ an asymptote meeting the curve at an infinitely distant point of inflexion.

Newton has given the following elegant construction for the description of this curve by continuous motion: A right angle has the side GF of fixed length, while the side GH passes through the fixed point E; a pencil at the middle point of GF will describe the cissoid. The proof we leave to the reader. (Lardner's Algebraic Geometry, pp. 196, 472.)



The cissoid is also the locus which we should find if we take on each of the radii vectores from the vertex of a parabola a portion equal to the reciprocal of its length. It is consequently also the locus of the foot of a perpendicular let fall from the vertex of a parabola on the tangent; or, in other words, if a parabola roll on an equal one, the locus of the vertex of the moving parabola will be the cissoid.

In all curves of the third class we have shown that there are three foci such that the product of the three perpendiculars on the tangent is in a given ratio to the perpendicular from the intersection of the focal tangents. In the cissoid, since the two imaginary points at infinity are on the curve, two of these foci coincide at the point $y=0$, $x=-r$; the other focus is $y=0$, $x=8r$; and the square of the perpendicular from the first, multiplied by that from the second, is in a constant ratio to that from the cusp.

Other focal properties of curves of the third class will be mentioned in Sect. VIII.

SECT. VII.—CUBICS OF THE FOURTH CLASS.

183. Methods such as have been applied in the last section to cubics of the third class may also, though with less facility, be applied to those of the fourth class. The general equation of such curves may always be written in the form

$$(A^2 \pm B^2)\, C = A^3,$$

where C is the tangent at the one real point of inflexion which the curve *must* have, and $A^2 \pm B^2$ are the tangents at the double point AB, the upper sign belonging to the case of a conjugate

point, and the lower to that of a node. Any point on the curve may be determined by the two equations $B = \mu A$, $(1 \pm \mu^2) C = A$.

Let it be required to find the relation between the μ's of three points which are in a right line. If we suppose the curve intersected by the line $C = aA + bB$, we have

$$aA^3 + bBA^2 \pm aAB^2 \pm bB^3 = A^3.$$

This equation solved for $B:A$ gives the μ's of the points where the line meets the curve, which are plainly connected by the relation

$$\mu\mu' + \mu'\mu'' + \mu''\mu = \pm 1.$$

If the line touch at a point of inflexion, $\mu = \mu' = \mu''$, and therefore $\mu^2 = \pm \frac{1}{3}$. Hence, *A cubic with a conjugate point has three real points of inflexion, a cubic with a node has one real and two imaginary.* This agrees with what we have seen in Sect. III.

The equation of the line joining two points will be found to be

$$(\mu + \mu') B \pm (1 \pm \mu^2)(1 \pm \mu'^2) C - (\mu^2 + \mu\mu' + \mu'^2 \pm 1) A = 0,$$

and therefore the equation of a tangent is

$$2\mu B \pm (1 \pm \mu^2)^2 C - (3\mu^2 \pm 1) A = 0,$$

or
$$C\mu^4 \pm (2C - 3A)\mu^2 \pm 2B\mu + C - A = 0;$$

whence can be readily inferred the relations between the μ's of four points whose tangents meet in a point. There is no difficulty in applying these equations to examples, but the subject does not seem of sufficient interest to be entitled to further space.

184. *Given the three points of inflexion and the tangents at them, the conjugate point must be the pole of the line joining the three points of inflexion, with regard to the triangle formed by the three tangents.*

Let the equation of the curve be

$$ABC = kD^3, \text{ where } D = A + B + C.$$

Then, since the double point must satisfy the equations

$$\frac{dS}{dA} = 0, \ \frac{dS}{dB} = 0, \ \frac{dS}{dC} = 0,$$

we must have

$$3kD^2 = AB = BC = CA.$$

The only point common to these is $A = B = C$, which (Art. 155) proves the theorem enunciated. We must then have $k = \frac{1}{27}$, and

the equation of a curve having a conjugate point and three real points of inflexion must be of the form

$$(A + B + C)^3 = 27ABC,$$

or, what is the same thing,

$$A^{\frac{1}{3}} + B^{\frac{1}{3}} + C^{\frac{1}{3}} = 0,$$

where A, B, C are the tangents at the points of inflexion.

185. As an example to illustrate cubics of the fourth class, we give *the caustic by reflexion of the parabola, the rays being perpendicular to the axis;* or what is the same thing, the envelope of a line drawn through any point on a parabola perpendicular to the focal radius vector. Let the parameter of the parabola be $4m$, and by the equation of the parabola the problem becomes, "to find the curve such that the perpendicular on the tangent expressed in terms of the angle which it makes with the axis shall be $p = \dfrac{m}{\cos^2 \frac{1}{2}\omega}$."

But by Art. 110 it at once appears that the equation of such a curve must be

$$\rho^{\frac{1}{3}} \cos \tfrac{1}{3}\omega = m^{\frac{1}{3}}, \text{ or } \rho \cos^3 \tfrac{1}{3}\omega = m.$$

We transform to x and y co-ordinates, by putting $4\cos^3 \frac{1}{3}\omega = \cos\omega + 3\cos\frac{1}{3}\omega$, and then cubing, and we get

$$27(x^2 + y^2)m = (4m - x)^3,$$

or

$$27y^2m = (m - x)(x + 8m)^2,$$

a curve which manifestly has a double point, and has a real point of inflexion at infinity on the axis of y, while the two circular points at infinity are imaginary points of inflexion.

SECT. VIII.—FOCAL PROPERTIES OF CIRCULAR CUBICS.

186. The focal properties of cubics in general have never been investigated. In this section I shall give an account of the case where two foci coincide, which will happen when the curve passes through the two circular imaginary points at infinity. Curves of this kind bear the same relation to cubics in general that circles bear to conics; for if the line joining two intersections of a conic and a cubic be projected to infinity, and the conic pro-

jected into a circle (*Conics*, p. 306), the cubic will be projected into a curve of this nature. I have already stated (p. 128) that the materials for this section have been contributed by Dr. Hart.

If ω and ω' be the two imaginary points at infinity, the intersection of the imaginary tangents *at* these points gives rise to a double focus, and the intersection of the two sets of tangents which can be drawn *from* these points gives rise to sixteen foci, four of which are real. It appears from Art. 158 that these sixteen foci lie on four circles. We shall principally occupy ourselves with the case when the four *real* foci lie on the same circle. In fact, at first we did not perceive that the four real foci were not necessarily on a circle, and we have, consequently, given but little investigation to the less interesting case when the real foci belong to different circles. When the four real foci are on a circle, the curve will be of the nature of those discussed (Art. 129), and the distances of any point of the curve from three of the foci will be connected by the relation $lr + ms = (l+m)t$.

187. Given the four foci, ABCD, the values of the coefficients l, m, can be found. We shall denote the distances of any point from these four points by $\alpha\beta\gamma\delta$, and the distances of the point O (see figure on next page) from these points by $abcd$; then we have seen (p. 126) that if $(l\alpha + m\gamma) = (l+m)\beta$, we must have

$$\frac{l^2}{\text{BCD}} + \frac{m^2}{\text{ABD}} = \frac{(l+m)^2}{\text{ACD}},$$

or

$$\frac{l^2}{c(b+d)} + \frac{m^2}{a(b+d)} = \frac{(l+m)^2}{(a+c)d}.$$

This gives us, when the four points are known, a quadratic to determine $l:m$. Hence *two cubics may be described having the four given points for foci.*

Clearing of fractions the equation just written, and remembering that $ac = bd$, it becomes

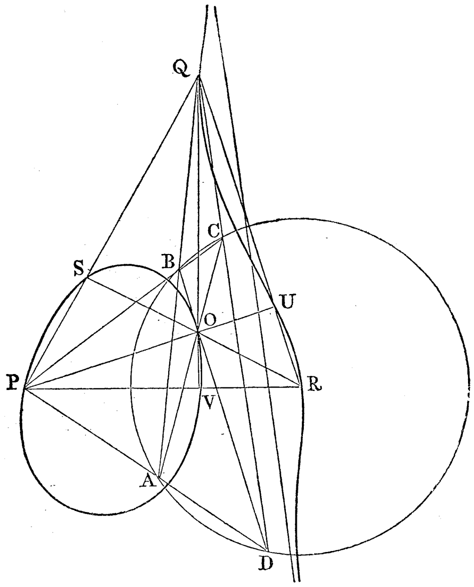
$$l^2(b^2-a^2) + 2lm(b^2+ac) + m^2(b^2-c^2) = 0,$$

which breaks up into the factors

$$l(b-a) + m(b+c) = 0, \quad l(b+a) + m(b-c) = 0.$$

The two curves, then, which have the given points for foci, have respectively the properties

$$(\mathrm{OB}+\mathrm{OC})\alpha \pm (\mathrm{OA}-\mathrm{OB})\gamma = \pm\, \mathrm{AC}.\beta, \quad (1)$$
$$(\mathrm{OC}-\mathrm{OB})\alpha \pm (\mathrm{OA}+\mathrm{OB})\gamma = \pm\, \mathrm{AC}.\beta. \quad (2)$$



When a relation of the form $l\alpha + m\gamma = n\beta$ is cleared of radicals it only involves the squares of lmn, therefore each of the coefficients must be understood as capable of a double sign. The different signs correspond to different branches of the same curve. The upper signs belong to a branch extending to infinity, for then the equation is satisfied by the values $\alpha = \beta = \gamma$, which must be true for an infinitely distant point. The lower signs belong to an oval, since then the equation is not satisfied by the values $\alpha = \beta = \gamma$. Since these values are also true for the centre of the circle, it appears that both curves intersect at the centre, and that the centre lies on the serpentine branches of both curves.

In like manner we must have the relations

$$(\mathrm{OC}-\mathrm{OD})\alpha \pm (\mathrm{OA}+\mathrm{OD})\gamma = \pm\, \mathrm{AC}.\delta, \quad (1)$$
$$(\mathrm{OC}+\mathrm{OD})\alpha \pm (\mathrm{OA}-\mathrm{OD})\gamma = \pm\, \mathrm{AC}.\delta. \quad (2)$$

If we subtract either the two equations (1), or the two equations (2), using in both cases the upper sign, we get

$$\frac{\alpha-\gamma}{\mathrm{AC}} = \frac{\beta-\delta}{\mathrm{BD}};$$

or *the locus of the intersections of two similar hyperbolæ, whose foci are* A *and* C, B *and* D, *consists of the serpentine branches of both curves.*

If we subtract, using the lower signs, we get

$$\frac{\alpha+\gamma}{\mathrm{AC}} = \frac{\beta+\delta}{\mathrm{BD}},$$

or, *the locus of the intersections of two similar ellipses having* A *and* C, B *and* D, *for foci, consists of the oval parts of both curves.* And in fact, it is plain, from the nature of the case, that this latter locus must be a finite curve; for when the axes major of both ellipses become infinite they cannot intersect unless the axes be equal to each other, while AC, BD, the distance between the foci, are not in general equal to each other.

It will be seen that both the equations (1), (2), using the lower signs, are satisfied for the point O, where the lines AC, BD intersect; hence the ovals of both curves intersect in this point. The same thing would appear from observing that this point satisfies both the conditions

$$\frac{\alpha+\beta}{AB}=\frac{\gamma+\delta}{CD},\quad \frac{\alpha+\delta}{AD}=\frac{\beta+\gamma}{BC}.$$

It can be proved in like manner that the points AD, BC; AB, CD are also points on both curves, and it will be found that these are points where the oval of one curve intersects the serpentine of the other.

188. We can at once perceive the direction of the tangents at any of the points O, P, Q. If, for instance, we consider the similar ellipses which intersect at O, and which have A, B; C, D for their foci, it is plain that they have a common tangent, namely, one of the bisectors of the angle at O; and obviously this common tangent must also be a tangent to the locus. Similarly for the points P, Q. Hence the two curves cut each other at right angles at the points O, P, Q, and the tangents are the bisectors of the angles at these points. These bisectors are parallel to each other (*Conics*, note, p. 245).

We shall now prove that any of these bisectors is parallel to the real asymptote of the curve to which it is a tangent. For, by the last Article, the direction of the point at infinity on the curve must be the same as that on two similar hyperbolæ, which have A, C; B, D for foci, and which have a common point at infinity; but since the hyperbolæ are similar, the common direction of the asymptotes must make equal angles with each axis major, it must therefore be parallel to the bisector of the angle at O.

Projecting this figure, we arrive at the theorem of Art. 135,

that if ABCD be the four points in which the tangents from any point ω meet those from another point of the curve ω', then these six points lie on the same conic; and OPQ, the intersections of diagonals and of opposite sides of the quadrilateral ABCD, are points on the curve, the tangents to which all pass through the point where $\omega\omega'$ meets the curve again.

189. We have seen that the centre of the circle is also a point on the locus, and we shall now prove that both curves intersect each other at right angles at that point, the directions of the tangents being parallel to those at the points OPQ. In general, it may be observed, that if we have a relation connecting the distances between any point on the curve and several fixed points, of the form

$$l\alpha + m\beta + n\gamma + \&c. = 0;$$

since, in passing to the consecutive point, we must have

$$ld\alpha + md\beta + nd\gamma + \&c. = 0;$$

and since each element of a focal radius vector is equal to the element of the arc multiplied by the cosine of the angle between that radius vector and the tangent, we must have

$$l\cos\mathrm{APT} + m\cos\mathrm{BPT} + n\cos\mathrm{CPT} + \&c. = 0.$$

Hence, if the centre be R, the relation

$$\frac{\alpha - \gamma}{\mathrm{AC}} = \frac{\beta - \delta}{\mathrm{BD}}$$

gives us

$$\frac{\cos\mathrm{ART} - \cos\mathrm{CRT}}{\mathrm{AC}} = \frac{\cos\mathrm{BRT} - \cos\mathrm{DRT}}{\mathrm{BD}};$$

and if we draw RE, RF, bisecting the angles ARC, BRD, this equation becomes

$$\frac{\sin\mathrm{ERT}\sin\tfrac{1}{2}\mathrm{ARC}}{\mathrm{AC}} = \frac{\sin\mathrm{FRT}\sin\tfrac{1}{2}\mathrm{BRD}}{\mathrm{BD}},$$

or simply ERT = FRT. The tangent then bisects the angle between the two lines RE, RF; but these are perpendicular to the lines AC, BD; the tangent is therefore parallel to one of the bisectors of the angle between the latter lines.

When the figure is projected, we learn from this theorem that the pole of the line $\omega\omega'$, with regard to the conic through $\omega\omega'$

ABCD, is also a point on the curve, the tangent at which passes through the point where $\omega\omega'$ meets the curve again.

190. Since OPQR are four points of contact of tangents from the same point of the curve, it follows, by Art. 133, that the point where OP meets QR (or the foot of the perpendicular from O on QR) is a point on the curve. Similarly for the point where OQ meets PR, and where OR meets PQ. It may, without much difficulty, be verified that these points satisfy the condition

$$\frac{\alpha \pm \gamma}{AC} = \frac{\beta \pm \delta}{BD},$$

and it can be proved that at these points also the curves cut at right angles. In fact, for one curve, S fulfils the condition

$$\frac{\alpha + \beta}{AB} = \frac{\gamma + \delta}{CD},$$

and therefore, if ST be the tangent at that point,

$$\frac{\cos AST + \cos BST}{AB} = \frac{\cos CST + \cos DST}{CD}.$$

Now the angles ASB, CSD may easily be seen to be equal, therefore, if SL, SM be the bisectors of these angles, this equation gives

$$\frac{\cos LST}{AB} = \frac{\cos MST}{CD}.$$

But S also lies on the curve

$$\frac{\alpha - \beta}{AB} = \frac{\delta - \gamma}{CD},$$

which gives, in like manner, for the tangent to the other curve,

$$\frac{\sin LST'}{AB} = \frac{\sin MST'}{CD}.$$

Hence, $\cos LST = \sin LST'$, and the two tangents are at right angles to each other. Hence, given the four foci, two cubics can be constructed, both passing through the points $\omega\omega'$, and cutting each other at right angles at seven other points OPQRSUV, all of which can easily be constructed.

It can be proved that the rectangle under the segments made by the cubic on any line through one of the points, OPQR, is equal to the rectangle under the segments made by the circle on

the same line; and hence that the tangents drawn from any of these points to the cubic are equal; and also, that tangents at the four points where the cubic meets the circle are diameters of the circle.

191. If two of the foci, C and D, coincide, then the intersections of AC, BD; AD, BC coincide with them, and form a double point on the curve. Its equation will then be

$$\text{BC}\alpha \pm \text{AC}\beta = \pm (\text{BC} \pm \text{AC})\gamma,$$

which is reducible to

$$\text{BC}(\alpha \pm \gamma) = \pm \text{AC}(\beta \pm \gamma);$$

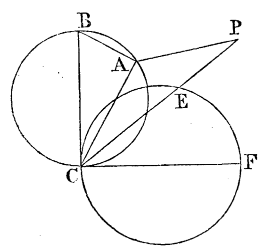
and the curve may, as before, be considered as the locus of the intersection of two similar conics with given foci, only that now one focus is common to both conics.

This mode of generation for a circular cubic of the fourth class leads without much difficulty to another. Equating the values of the radius vector of the two similar conics which have C for a common focus, we get, if P be any point of the curve, and if AC be put $= a$, BC $= b$,

$$\frac{1 - e\cos \text{PCA}}{a} = \frac{1 - e\cos \text{PCB}}{b}, \quad e = \frac{a - b}{a\cos \text{PCB} - b\cos \text{PCA}}.$$

If we draw a line CF, and put

$$\text{PCB} = \text{BCF} - \text{PCF},$$
$$\text{PCA} = \text{ACF} - \text{PCF},$$



this will become

$$e = \frac{a - b}{(a\cos \text{BCF} - b\cos \text{ACF}) \cos \text{PCF}},$$

provided we have $b\sin \text{ACF} = a\sin \text{BCF}$, that is, provided CF be drawn to touch the circle which circumscribes the triangle ABC.

Now
$$\text{PC} \pm \text{PA} = \frac{a}{e} = \frac{a\cos \text{BCF} - b\cos \text{ACF}}{a - b} a\cos \text{PCF}.$$

If, therefore, we describe a circle whose diameter is

$$\text{CF} = \frac{a\cos \text{BCF} - b\cos \text{ACF}}{a - b} a,$$

then the right-hand side of the preceding equation will represent

CE, and we have PA = PE. A similar equation may be had for PB, taking $CF' = \frac{b}{a} CF$. Hence, *a circular cubic of the fourth class may be considered as the locus of a point whose distance from a fixed point is equal to its distance from a fixed circle, the latter distance being measured on the radius vector to a fixed point of the circle.*

This mode of generation is quite analogous to the manner in which the common parabola is generated by focus and directrix.

192. It is manifest that in this mode of generation the two tangents at the double point are the two chords of the circle which are equal to AC. If, therefore, AC be less than CF, these chords will be real; if it be greater, these chords will be imaginary, and C will be a conjugate point. If AC = CF, the two chords coincide, and the double point will be a cusp. A third focus, B, will coincide with C and D at the cusp; and this mode of generation is more important, since the equation $l\beta + m\gamma = (l + m)\delta$ becomes illusory when the points BCD coincide. The direction of the asymptote of the curve will be parallel to the bisector of the angle ACF. In each case the direction of the asymptote is evidently perpendicular to FA.

193. It was proved (Art. 128) that in curves of the third class if $\alpha\beta\gamma$ be the perpendiculars from the three foci on any tangent, the tangents from these foci meet in a point δ, and

$$\alpha\beta\gamma = k\delta.$$

It follows, as we pass to the consecutive point, that

$$\frac{d\alpha}{\alpha} + \frac{d\beta}{\beta} + \frac{d\gamma}{\gamma} - \frac{d\delta}{\delta} = 0.$$

But if AP be the perpendicular on the tangent at any point R of the curve, and $d\theta$ the angle between two consecutive tangents, we have $d\alpha = RPd\theta$, and $\frac{RP}{AP} = \cot ARP$; hence the relation is immediately inferred,

$$\cot ART + \cot BRT + \cot CRT = \cot DRT;$$

or, *the sum of the cotangents of the angles made with the tangent by the radii vectores from the three foci, is equal to the cotangent of the*

angle made by the radius vector from the point where the focal tangents intersect.

In circular cubics two of the foci coincide, the equation becomes $a\gamma^2 = k\delta$, and the point δ is the point of contact of the tangent from the double focus γ.

194. In investigating the properties of circular cubics, it is often convenient to use the equation

$$\rho^2 A = k^2 B,$$

where ρ expresses the distance from the double focus, A is the real asymptote, and B is a line passing through the finite point, where the asymptote meets the curve, and also through the imaginary points, where the two imaginary asymptotes meet the curve again. Since any line drawn through the point AB obviously meets the curve in two points equidistant from the double focus, it follows that the point of contact of any tangent from AB must be the foot of the perpendicular from the double focus, and also, that if the curve have a double point, the lines joining it to AB and to the double focus are at right angles.

Dr. Hart finds the condition that the curve should have a double point or a cusp, by forming the equation of the tangents from the point AB, and applying the condition that two of them, or that all three should coincide. He finds thus: if the line joining AB to the double focus make angles a, β with A, B, and an angle θ with the line joining AB to the cusp,

$$\beta = 3\theta, \ \tan\theta = 3\tan a.$$

In this case the cuspidal tangent bisects the angle between the asymptote and the line joining AB to the cusp, and touches the circle passing through the cusp, double focus, and AB.

195. In all that precedes we have gone on the supposition that the four real foci lie on a circle. Since, however, it was proved (Art. 188) that the point O in this method of generation always belongs to a real oval, and since we saw (Art. 149) that there are some cubics which have no real ovals, it is plain that this mode of generation does not apply to all cubics, and therefore that the theorem of p. 127 has been too unguardedly stated. The argument I there relied on was, that the equation $lr + ms = (l + m)t$,

containing seven constants, expressed or implied, was the most general equation of a circular cubic; but the argument fails, because though only a determinate number of cubics can be described having six tangents and three points given, there still may be more than one: and though we can describe a circular cubic fulfilling the conditions $lr + ms = (l + m)t$, and having three foci and one point common with any given circular cubic, it does not necessarily follow that the two curves must coincide.

Let us now examine what becomes of the equation

$$\frac{\alpha \pm \beta}{AB} = \frac{\gamma \pm \delta}{CD},$$

when the points CD are the imaginary points in which a given line meets a certain circle through the points AB. When the imaginary foci of a conic are given, its real foci are also given, and lie on a line perpendicular to the line joining the imaginary foci, and at a distance from each other, equal to $CD\sqrt{-1}$. If, therefore, M be the foot of the perpendicular on CD from the centre of the circle, and if we take $EM = FM = MT$, where T is the point of contact of the tangent from M, then the given cubic may be considered as the locus of intersections of a conic having AB for foci, with a certain other conic having given points EF for real foci. But the sum of the distances $\gamma + \delta$ from the imaginary foci is twice the axis *minor*, or conjugate axis, of the second conic; if then ϵ, ϕ denote the distances from E, F, the real foci, we have

$$\frac{(\alpha \pm \beta)^2}{AB^2} + \frac{(\epsilon \pm \phi)^2}{EF^2} = 1.$$

This is the mode of generation to which we are led for cubics of this nature; it does not appear so fertile of consequences as that for the other species of circular cubics, but, as I have already mentioned, this class of curves escaped our notice at first, and has not received much attention.

SECT. IX.—GENERAL EQUATION OF THE THIRD DEGREE.

196. We have left for this section all discussions which require the use of the general equation of the curve, and shall commence with the formation of the function H(U), (p. 71).

The general equation being written as at page 99, we have, suppressing the factor 6 common to all the coefficients:

$$\frac{d^2U}{dx^2} = a_1x + a_2y + a_3z; \quad \frac{d^2U}{dydz} = dx + b_3y + c_1z;$$

$$\frac{d^2U}{dy^2} = b_1x + b_2y + b_3z; \quad \frac{d^2U}{dzdx} = a_3x + dy + c_1z;$$

$$\frac{d^2U}{dz^2} = c_1x + c_2y + c_3z; \quad \frac{d^2U}{dxdy} = a_2x + b_1y + dz.$$

The coefficients in H are then as follows:

$$\mathrm{a}_1 = a_1d^2 + b_1a_3^2 + c_1a_2^2 - a_1b_1c_1 - 2da_2a_3.$$

$$\mathrm{b}_2 = b_2d^2 + c_2b_1^2 + a_2b_3^2 - a_2b_2c_2 - 2db_1b_3.$$

$$\mathrm{c}_3 = c_3d^2 + a_3c_2^2 + b_3c_1^2 - a_3b_3c_3 - 2dc_1c_2.$$

$$3\mathrm{a}_2 = c_2a_2^2 + a_2c_1b_1 - 2a_2a_3b_3 - a_2d^2 + 2a_1b_3d + b_2a_3^2 - a_1c_1b_2 - a_1b_1c_2.$$

$$3\mathrm{b}_1 = c_1b_1^2 + b_1c_2a_2 - 2b_1b_3a_3 - b_1d^2 + 2a_3b_2d + a_1b_3^2 - a_1b_2c_2 - a_2b_2c_1.$$

$$3\mathrm{a}_3 = b_3a_3^2 + a_3b_1c_1 - 2a_3a_2c_2 - a_3d^2 + 2a_1c_2d + c_3a_2^2 - a_1b_1c_3 - a_1c_1b_3.$$

$$3\mathrm{c}_1 = b_1c_1^2 + c_1a_3b_3 - 2c_1a_2c_2 - c_1d^2 + 2a_2c_3d + a_1c_2^2 - a_1b_3c_3 - a_3c_3b_1.$$

$$3\mathrm{b}_3 = a_3b_3^2 + b_3a_2c_2 - 2b_3b_1c_1 - b_3d^2 + 2b_2c_1d + c_3b_1^2 - b_2c_3a_2 - b_2c_2a_3.$$

$$3\mathrm{c}_2 = a_2c_2^2 + c_2a_3b_3 - 2c_2b_1c_1 - c_2d^2 + 2b_1c_3d + b_2c_1^2 - b_2c_3a_3 - c_3b_3a_2.$$

$$6\mathrm{d} = 2d(b_1c_1 + c_2a_2 + a_3b_3) - 2d^3 + (a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1) - a_1b_2c_3 - 3(a_2b_3c_1 + a_3b_1c_2).$$

197. By the help of this equation we can readily find the conditions which must be fulfilled in order that the general equation should represent three right lines. For, when the equation represents three right lines, since the line joining any two consecutive points of the system passes also through the next consecutive point, the condition H must be satisfied for *every* point of the system; and therefore the equation $H = 0$ must represent the same three lines denoted by U. We have therefore the conditions

$$\frac{\mathrm{a}_1}{a_1} = \frac{\mathrm{a}_2}{a_2} = \frac{\mathrm{a}_3}{a_3} = \frac{\mathrm{b}_1}{b_1} = \frac{\mathrm{b}_2}{b_2} = \frac{\mathrm{b}_3}{b_3} = \frac{\mathrm{c}_1}{c_1} = \frac{\mathrm{c}_2}{c_2} = \frac{\mathrm{c}_3}{c_3} = \frac{\mathrm{d}}{d},$$

a system of forty-five equations, which at first sight appear to be equivalent to nine, but which can only be equivalent to three independent equations. For (*Conics*, p. 70) only three conditions are necessary in order that the equation of the third degree, containing nine constants, should reduce to a system of three right lines, which involves six constants. The reader may ve-

rify that the preceding equations actually do reduce to three, by the help of the following equations, which are identically true, and of the other equations of similar form, which may be obtained by interchange of letters:

$$(\mathrm{a}_1 b_2 - \mathrm{b}_2 a_1) = 3\,(\mathrm{a}_2 b_1 - \mathrm{b}_1 a_2).$$
$$2\,(\mathrm{a}_2 d - a_2 \mathrm{d}) + (\mathrm{a}_3 b_1 - \mathrm{b}_1 a_3) + (\mathrm{b}_3 a_1 - \mathrm{a}_1 b_3) = 0.$$
$$(\mathrm{b}_1 c_1 - \mathrm{c}_1 b_1) + (\mathrm{c}_2 a_2 - \mathrm{a}_2 c_2) + (\mathrm{a}_3 b_3 - \mathrm{b}_3 a_3) = 0.$$

198. It has been proved (pp. 75, 142) that if we form the Hessian of an equation of the form $l\mathrm{H} + m\mathrm{U}$, then $\mathrm{H}(l\mathrm{H} + m\mathrm{U})$ must also be of the form $\lambda\mathrm{H} + \mu\mathrm{U}$. We proceed to form the actual values of the new coefficients, and commence, for simplicity, with the equation to which we have seen (p. 136) that every equation of the third degree is reducible,

$$\mathrm{U} = a_1x^3 + b_2y^3 + c_3z^3 + 6dxyz,$$

and we find

$$\mathrm{HU} = d^2(a_1x^3 + b_2y^3 + c_3z^3) - (a_1b_2c_3 + 2d^3)\,xyz.$$

We have suppressed in H the factor 216, arising from the six in each differential coefficient. We have then

$$l\mathrm{U} + m\mathrm{H} = (l + md^2)\;(a_1x^3 + b_2y^3 + c_3z^3) + (6ld - ma_1b_2c_3 - 2md^3)xyz\,;$$

and if we form in like manner $\mathrm{H}(l\mathrm{U} + m\mathrm{H})$, we shall find it to be

$$3\,(l + md^2)\;(6ld - ma_1b_2c_3 - 2md^3)^2\;(a_1x^3 + b_2y^3 + c_3z^3)$$
$$- 108\,(l + md^2)^3\; a_1b_2c_3xyz - (6ld - ma_1b_2c_3 - 2md^3)^3\,xyz.$$

To reduce this to the form $\lambda\mathrm{U} + \mu\mathrm{H}$ we get the equations

$$\lambda + \mu d^2 = 3\,(l + md^2)\;(6ld - ma_1b_2c_3 - 2md^3)^2.$$
$$6\lambda d - \mu(a_1b_2c_3 + 2d^3) = -108\,(l + md^2)^3\,a_1b_2c_3 - (6ld - ma_1b_2c_3 - 2md^3)^3.$$

Solving these equations it will be found, if we write

$$\mathrm{S} = d^4 - da_1b_2c_3\,;\quad \mathrm{T} = a_1^2b_2^2c_3^2 - 20a_1b_2c_3d^3 - 8d^6.$$
$$\lambda = 36\mathrm{S}l^2m + 3\mathrm{T}lm^2 + 4\mathrm{S}^2m^3\,;\quad \mu = 108l^3 - 36\mathrm{S}lm^2 - \mathrm{T}m^3.$$

Now it is plain from the nature of the case, that if the reduced equation which we have used were made by transformation of coordinates to assume the general form, the form of the values of λ, μ must remain the same, only that S and T would now be what the functions we have given become on transformation of coordinates.

Since, as a particular case of the foregoing, the second Hessian,

$$H(HU) = 4S^2U - TH,$$

it follows that the equation $T = 0$ will in general express the condition that the Hessian should be the original curve. If $S = 0$, the Hessian of the Hessian is the Hessian itself, and therefore that Hessian, by Art 197, breaks up into three right lines; if then we set aside the case when the curve U is a system of three right lines, $S = 0$ expresses the condition that the given equation should be reducible to a sum of three cubes, as readily appears from an examination of the value of S in the particular case hitherto discussed.

199. We proceed then to find the values of S and T in general. It is plain that the coefficients of any of the terms of $H(lU + mH)$ (for example A_1) are found by substituting in the value given for a_1 (Art. 196) $a_1 = la_1 + m\mathrm{a}_1$, $b_1 = lb_1 + m\mathrm{b}_1$, &c. And if we have so formed any pair of terms, since

$$A_1 = \lambda a_1 + \mu \mathrm{a}_1, \quad B_2 = \lambda b_2 + \mu \mathrm{b}_2,$$

we have

$$\lambda = \frac{A_1\mathrm{b}_2 - B_2\mathrm{a}_1}{a_1\mathrm{b}_2 - b_2\mathrm{a}_1}, \quad \mu = \frac{A_1 b_2 - B_2 a_1}{\mathrm{a}_1 b_2 - \mathrm{b}_2 a_1}.$$

Now it is obvious that the coefficient of l^3 in the value of A_1 is a_1, and the coefficient of l^3 in the value of B_2 is b_2. Hence the coefficient of l^3 in the value of λ is $= 0$, and in that of μ is $= 1$. We go on then to examine the coefficient of l^2m in A_1. Making the above substitution in the value given for a_1 (Art. 196), we obtain for the part multiplied by l^2m,

$$a_1(2d\mathrm{d} - b_1\mathrm{c}_1 - \mathrm{b}_1c_1) + \mathrm{a}_1(d^2 - b_1c_1) + \mathrm{b}_1a_3^2 + \mathrm{c}_1a_2^2$$
$$+ 2b_1a_3\mathrm{a}_3 + 2c_1a_2\mathrm{a}_2 - 2a_2d\mathrm{a}_3 - 2a_3d\mathrm{a}_2 - 2a_2a_3\mathrm{d}.$$

By the help of the identical equations of Art. 197, this becomes

$$a_1(2d\mathrm{d} - b_1\mathrm{c}_1 - \mathrm{b}_1c_1 + a_3\mathrm{b}_3 + a_2\mathrm{c}_2) + \mathrm{a}_1(d^2 - b_1c_1 - c_2a_2 - a_3b_3)$$
$$+ 3b_1a_3\mathrm{a}_3 + 3a_2c_1\mathrm{a}_2 - 6a_2a_3\mathrm{d}.$$

And substituting then the actual values of $\mathrm{a}_1, \mathrm{a}_2$ &c., it reduces to $\frac{1}{3}a_1S$, where

$$\begin{aligned} S = {} & d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2) - da_1b_2c_3 \\ & + d(a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1) - (b_3c_2a_2a_3 + a_3c_1b_1b_3 + a_2b_1c_1c_2) \\ & + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) - (a_1b_2c_1c_2 + b_2c_3a_2a_3 + c_3a_1b_3b_1) \\ & + (b_3c_3a_2^2 + b_2c_2a_3^2 + a_3c_3b_1^2 + a_1c_1b_3^2 + a_2b_2c_1^2 + a_1b_1c_2^2). \end{aligned}$$

In like manner the coefficient of l^2m in B_2 is $\frac{1}{3}b_2\mathrm{S}$; we have then the coefficient of l^2m in λ, $=\frac{1}{3}\mathrm{S}$, while the coefficient of l^2m in μ vanishes. These values agree with those obtained in Art. 198, only that we are to suppose the whole equation divided by 108.

It would have been, perhaps, more convenient in calculating S to use the coefficient D, for though it consists of several more terms than A_1 or B_2, yet its symmetry enables us to proceed more rapidly. For brevity I shall write $(b_1c_1 + ..)$ for $b_1c_1 + c_2a_2 + a_3b_3$; and in like manner in other cases, where one of a set of terms being given, the rest can be written down from symmetry. We have then the coefficient of l^2m in $6\mathrm{D} =$

$$2\mathrm{d}(b_1c_1 + ..) + 2d(b_1\mathrm{c}_1 + \mathrm{b}_1c_1 + ..) - 6d^2\mathrm{d} + (\mathrm{a}_1b_3c_2 + ..) - (\mathrm{a}_1b_2c_3 + ..)$$
$$+ (\mathrm{a}_2b_1c_3 + ..) - 3(\mathrm{a}_2b_3c_1 + ..)$$

which, by the help of the identical equations of Art. 197, becomes

$$6\mathrm{D} = 6\mathrm{d}(b_1c_1 + ..) - 6\mathrm{d}d^2 + 3(\mathrm{a}_1b_3c_2 + ..) - (\mathrm{a}_1b_2c_3 + ..) - 3(\mathrm{a}_2b_3c_1 + ..),$$

and putting in the values for a_1 &c., the right-hand side becomes $2\mathrm{S}d$, where S has the same value as that found already.

200. In precisely the same manner we may proceed to find the quantity multiplied by lm^2 in any of the coefficients. In D, for example, we have only to interchange Romans and Italics in the value given in the last Article, and we have

$$6\mathrm{D} = 6d(\mathrm{b}_1\mathrm{c}_1 + ..) - 6d\mathrm{d}^2 + 3(a_1\mathrm{b}_3\mathrm{c}_2 + ..) - (a_1\mathrm{b}_2\mathrm{c}_3 + ..) - 3(a_2\mathrm{b}_3\mathrm{c}_1 + ..)$$

It would, no doubt, be extremely difficult to reduce this to the form $\lambda d + \mu\mathrm{d}$, only that we have been led by Art. 198 to foresee that the result must be of the form $\frac{1}{6}\mathrm{T}d - 2\mathrm{Sd}$, where the value of S is that given in the last Article. If, therefore, we add 2Sd to the right-hand side of the equation, we can verify that it becomes divisible by d, and, performing this division, can obtain the value of T.

I give so much of the work as is necessary for the actual calculation of T:

$$36(\mathrm{b}_1\mathrm{c}_1 + ..) = 4(b_1c_1 + ..)d^4 - 16d^3(a_1b_3c_2 + ..) + 24d^2(a_1b_2c_1c_2 + ..)$$
$$- 8d^2(b_1^2c_1^2 + ..) + 8d^2(b_1c_1a_2c_2 + ..) + 16(c_3b_1c_2a_2^2 + ..)d + 4(b_1^3c_1^3 + ..)$$
$$- 32(a_1b_1b_3c_1c_2 + ..)d - 16a_1b_2c_3(b_1c_1 + ..)d - 12(b_1^2c_1^2a_2c_2 + ..)$$
$$+ 60a_2a_3b_1b_3c_1c_2 + 12(a_1b_1c_1^2b_3^2 + ..) - 12(a_1b_1a_3b_3c_2^2 + ..)$$
$$+ 4(a_1^2b_3^2c_2^2 + ..) - 4(a_1b_2c_1c_2a_3b_3 + ..) + 12a_1b_2c_3(b_3c_1a_2 + b_1c_2a_3)$$
$$- 4(a_1^2b_2c_2^3 + ..) + 4a_1b_2c_3(a_1c_2b_3 + ..).$$

$$-36\mathrm{d}^2 = -4d^6 + 8(b_1c_1 + \,.\,.)\,d^4 - 4a_1b_2c_3d^3 + 4d^3(a_1b_3c_2 + \,.\,.)$$
$$- 12d^3(a_2b_3c_1 + \,.\,.) - 4d^2(b_1^2c_1^2 + \,.\,.) - 8(b_1c_1a_2c_2 + \,.\,.)\,d^2$$
$$+ 4a_1b_2c_3(b_1c_1 + \,.\,.)d + 12(a_2b_3c_1 + \,.\,.)\,(b_1c_1 + \,.\,.)d - 9(a_2^2b_3^2c_1^2 + \,.\,.)$$
$$- 4(a_1b_1b_3c_1c_2 + \,.\,.)d - 4(c_3a_2^2b_1c_2)d - 18a_2a_3b_1b_3c_1c_2 - (a_1^2b_3^2c_2^2 + \,.\,.)$$
$$+ 6(a_1a_3b_1b_3c_2^2 + \,.\,.) - 2(a_1b_2c_1c_2a_3b_3 + \,.\,.) - 6a_1b_2c_3(a_2b_3c_1 + \,.\,.)$$
$$+ 2a_1b_2c_3(a_1b_3c_2 + \,.\,.) - a_1^2b_2^2c_3^2.$$

$$\frac{18(a_1\mathrm{b}_3\mathrm{c}_2 + \,.\,.)}{d} = 2d^3(a_1b_3c_2 + \,.\,.) - 8d^2(a_1b_2c_1c_2 + \,.\,.) - 4(c_3a_2^2b_1c_2)\,d$$
$$+ 4(a_1b_1b_3c_1c_2)\,d + 12a_1b_2c_3(b_1c_1 + \,.\,.)\,d - 12a_1b_2c_3(a_2c_1b_3 + \,.\,.)$$
$$- 4(a_1b_2c_1a_2c_2^2 + \,.\,.) + 8(a_1b_2c_1c_2a_3b_3 + \,.\,.) - 8a_1b_2c_3(a_1b_3c_2 + \,.\,.)$$
$$+ 4(a_1b_2^2c_1^3 + \,.\,.) + \,.\,.$$

$$\frac{-18(a_2\mathrm{b}_3\mathrm{c}_1 + \,.\,.)}{d} = -6d^3(a_2b_3c_1 + \,.\,.) + 8(b_3c_3a_2^2 + \,.\,.)d^2 + 4(c_3a_2^2b_1c_2 + \,.\,.)d$$
$$- 4(a_1b_1b_3c_1c_2 + \,.\,.)\,d - 12(b_2c_3c_1a_2^2 + \,.\,.)\,d + 4(a_1b_1c_1^2b_3^2 + \,.\,.)$$
$$- 8(a_1b_1a_2c_2^3 + \,.\,.) + 4(a_1a_3b_1b_3c_2^2 + \,.\,.) - 8(a_1^2b_3^2c_2^2 + \,.\,.)$$
$$+ 8(a_1b_2c_1c_2a_3b_3 + \,.\,.) + 12a_1b_2c_3(a_2c_1b_3 + \,.\,.) + 4(a_1b_2^2c_1^3) + \,.\,.$$

$$\frac{-6\,(a_1\mathrm{b}_2\mathrm{c}_3 + \,.\,.)}{d} = -18a_1b_2c_3d^3 + 24(a_1b_2c_1c_2 + \,.\,.)d^2 - 12(b_2c_3c_1a_2^2 + \,.\,.)$$
$$+ 12a_1b_2c_3(b_1c_1 + \,.\,.)d - 24(a_1b_1b_3c_1c_2 + \,.\,.) + 12(a_1b_1a_3b_3c_2^2 + \,.\,.)$$
$$+ 12(a_1b_1b_3^2c_1^2 + \,.\,.) - 12(a_1b_2a_2c_1c_2^2 + \,.\,.) + \,.\,.$$

$$\frac{12\mathrm{Sd}}{d} = -4d^6 + 12(b_1c_1 + \,.\,.)\,d^4 + 2a_1b_2c_3d^3 - 2d^3(a_1b_3c_2 + \,.\,.)$$
$$- 18d^3(a_2b_3c_1 + \,.\,.) - 12d^2(b_1^2c_1^2 + \,.\,.) - 12d^2(b_1c_1a_2c_2 + \,.\,.)$$
$$+ 4d^2(b_3c_3a_2^2 + \,.\,.) - 4d^2(a_1b_2c_1c_2 + \,.\,.) + 24d(a_2b_3c_1 + \,.\,.)\,(b_1c_1 + \,.\,.)$$
$$+ 4(b_1^3c_1^3 + \,.\,.) - 18(a_2^2b_3^2c_1^2 + \,.\,.) - 48a_2a_3b_1b_3c_1c_2 - 4(a_1b_1c_1^2b_3^2 + \,.\,.)$$
$$- 4(a_1b_1a_2c_2^3 + \,.\,.) - 4(a_1a_3b_1b_3c_2^2 + \,.\,.) + 2\,(a_1^2b_3^2c_2^2 + \,.\,.)$$
$$+ 4(a_1b_2b_1c_2c_1^2 + \,.\,.) + 8(a_1b_2c_1c_2a_3b_3 + \,.\,.) - 4a_1b_2c_3\,(a_1b_3c_2 + \,.\,.)$$
$$+ 2a_1^2b_2^2c_3^2 + \,.\,.$$

Assembling the terms, and writing at full length, we find

$$\mathrm{T} = -8d^6 + 24d^4(b_1c_1 + c_2a_2 + a_3b_3) - 20a_1b_2c_3d^3 - 36d^3(a_2b_3c_1 + a_3b_1c_2)$$
$$- 12d^3(a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1) - 12d^2(b_1c_1c_2a_2 + c_2a_2a_3b_3 + a_3b_3b_1c_1)$$
$$- 24d^2(b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) + 36d^2(a_1b_2c_1c_2 + b_2c_3a_2a_3 + c_3a_1b_3b_1)$$
$$+ 12d^2(b_3c_3a_2^2 + b_2c_2a_3^2 + a_3c_3b_1^2 + a_1c_1b_3^2 + a_2b_2c_1^2 + a_1b_1c_2^2)$$
$$+ 36d(a_2b_3c_1 + a_3b_1c_2)\;(b_1c_1 + c_2a_2 + a_3b_3)$$
$$+ 12d(a_1a_2b_3c_2^2 + a_1a_3c_2b_3^2 + b_2b_3c_1a_3^2 + b_2b_1a_3c_1^2 + c_3c_1a_2b_1^2 + c_3c_2b_1a_2^2)$$
$$- 60d(a_1b_1b_3c_1c_2 + b_2c_1c_3a_2a_3 + c_3a_2a_3b_1b_3)$$
$$- 24d(a_1b_2a_3c_2^2 + a_1b_2b_3c_1^2 + b_2c_3b_1a_3^2 + b_2c_3c_1a_2^2 + c_3a_1c_2b_1^2 + c_3a_1a_2b_3^2)$$

$$
\begin{aligned}
&+ 12da_1b_2c_3(b_1c_1 + c_2a_2 + a_3b_3) + 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3) \\
&- 27(a_2^2b_3^3c_1^2 + a_3^2b_1^2c_2^2) - 6b_1c_1c_2a_2a_3b_3 \\
&- 12(b_1^2c_1^2c_2a_2 + b_1^2c_1^2a_3b_3 + c_2^2a_2^2a_3b_3 + c_2^2a_2^2b_1c_1 + a_3^2b_3^2b_1c_1 + a_3^2b_3^2c_2a_2) \\
&+ 24(a_1b_1b_3^2c_1^2 + a_1c_1c_2^2b_1^2 + b_2c_2c_1^2a_2^2 + b_2a_2a_3^2c_2^2 + c_3a_3a_2^2b_3^2 + c_3b_3b_1^2a_3^2) \\
&- 12(a_1a_2b_1c_2^3 + a_1a_3c_1b_3^3 + b_2b_3c_2a_3^3 + b_2b_1a_2c_1^3 + c_3c_1a_3b_1^3 + c_3c_2b_3a_2^3) \\
&+ 6(a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1)\ (a_2b_3c_1 + a_3b_1c_2) \\
&- 3(a_1^2b_3^2c_2^2 + b_2^2c_1^2a_3^2 + c_3^2a_2^2b_1^2) \\
&+ 18(a_1b_2a_3b_3c_1c_2 + b_2c_3b_1c_1a_2a_3 + c_3a_1c_2a_2b_3b_1) - 12(a_1b_2a_2c_1c_2^2 \\
&+ a_1b_2b_1c_2c_1^2 + b_2c_3c_2a_3a_2^2 + b_2c_3b_3a_2a_3^2 + c_3a_1c_1b_3b_1^2 + c_3a_1a_3b_1b_3^2) \\
&+ 4(a_1^2b_2c_2^3 + a_1^2c_3b_3^3 + b_2^2c_3a_3^3 + b_2^2a_1c_1^3 + c_3^2a_1b_1^3 + c_3^2b_2a_2^3) \\
&+ 6a_1b_2c_3(a_2b_3c_1 + a_3b_1c_2) - 6a_1b_2c_3(a_1b_3c_2 + b_2c_1a_3 + c_3a_2b_1) + a_1^2b_2^2c_3^2.
\end{aligned}
$$

In like manner can be calculated the quantity multiplied by m^3 in any of the coefficients. I have thought it needless to take the trouble of verifying that the coefficient of m^3 in D, for example, is $\frac{1}{27}S^2d - \frac{1}{108}Td$.

201. By the help of the preceding Articles we can obtain the equation of any of the systems of three right lines which pass through the nine points of inflexion. This problem is equivalent to determining $\lambda : \mu$ so that $\lambda U + \mu H$ shall represent three right lines. But when this is the case the same system must also be represented by the Hessian of $\lambda U + \mu H$, or by

$$(36S\lambda^2\mu + 3T\lambda\mu^2 + 4S^2\mu^3)\,U + (108\lambda^3 - 36S\lambda\mu^2 - T\mu^3)\,H.$$

$\lambda : \mu$ is then determined by the equation

$$\frac{\lambda}{\mu} = \frac{36S\lambda^2\mu + 3T\lambda\mu^2 + 4S^2\mu^3}{108\lambda^3 - 36S\lambda\mu^2 - T\mu^3},$$

or

$$27\lambda^4 - 18S\lambda^2\mu^2 - T\lambda\mu^3 - S^2\mu^4 = 0.$$

Now this equation, treated by the ordinary methods (see Lacroix, Complément, § 20), gives

$$2\frac{\lambda}{\mu} = \sqrt{z'} \pm \sqrt{z''} \pm \sqrt{z'''},$$

where z', z'', z''' are the roots of the equation

$$729z^3 - 972Sz^2 + 432S^2z - T^2 = 0,$$

or

$$(9z - 4S)^3 = T^2 - 64S^3.$$

To each of the four values of $\lambda : \mu$ thus obtained corresponds one

of the four systems of right lines, which we have seen (Art. 147) may be drawn through the nine points of inflexion.

202. We proceed now to calculate the equation $R = 0$, which expresses the condition that the general equation of the third degree should represent a cubic with a double point. This is to eliminate the variables between the three equations

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0.$$

M. Hesse has given the following method of performing this elimination, which may also be extended so as to solve the problem of eliminating the variables between any three equations of the second degree.

We have seen (Art. 84) that any double point on the curve U is also a double point on the curve H; it must, therefore, also satisfy the equations

$$\frac{dH}{dx} = 0, \quad \frac{dH}{dy} = 0, \quad \frac{dH}{dz} = 0.$$

Now between these six homogeneous equations, which are satisfied for the double points, we can eliminate the six quantities $x^2, y^2, z^2, xy, yz, zx$, as if they were independent variables, and the equations linear functions of them. The required result is therefore the following determinant:

$$R = \begin{vmatrix} a_1, & b_1, & c_1, & a_2, & a_3, & d, \\ a_2, & b_2, & c_2, & b_1, & d, & b_3, \\ a_3, & b_3, & c_3, & d, & c_1, & c_2, \\ \mathrm{a}_1, & \mathrm{b}_1, & \mathrm{c}_1, & \mathrm{a}_2, & \mathrm{a}_3, & \mathrm{d}, \\ \mathrm{a}_2, & \mathrm{b}_2, & \mathrm{c}_2, & \mathrm{b}_1, & \mathrm{d}, & \mathrm{b}_3, \\ \mathrm{a}_3, & \mathrm{b}_3, & \mathrm{c}_3, & \mathrm{d}, & \mathrm{c}_1, & \mathrm{c}_2. \end{vmatrix}$$

The actual calculation, however, of this determinant in terms of the coefficients of the original equation is a work of no small labour; and I have found it more convenient in practice to use the old-fashioned method of elimination. We substitute in $\frac{dU}{dz}$ successively the co-ordinates of each of the intersections of $\frac{dU}{dx}, \frac{dU}{dy}$: multiply the four results together, and then express the sym-

metric functions of the co-ordinates in terms of the coefficients of $\frac{d\text{U}}{dx}, \frac{d\text{U}}{dy}$. By this method I had made considerable progress in working out the result, when M. Aronhold's publication of the form of R suggested to me a shorter process.

203. If we were to form the equation of the four tangents drawn from any point of a cubic to the curve, it is easy to see that the only case when this equation can have two equal factors is when the curve has a double point, since a cubic has no double tangents. Let us now suppose, in the first place, that the origin is on the curve, and $c_3 = 0$. The equation of the curve is

$$(a_1x^3 + 3a_2x^2y + 3b_1xy^2 + b_2y^3) + 3(a_3x^2 + 2dxy + b_3y^2)z + 3(c_1x + c_2y)z^2 = 0.$$

And the tangents drawn from the origin are represented (Art. 79) by the equation

$$3(a_3x^2 + 2dxy + b_3y^2)^2 = 4(c_1x + c_2y)(a_1x^3 + 3a_2x^2y + 3b_1xy^2 + b_2y^3),$$

or, $$(3a_3^2 - 4a_1c_1)x^4 + 6(a_3b_3 + 2d^2 - 2b_1c_1 - 2a_2c_2)x^2y^2 +$$
$$4(3a_3d - 3a_2c_1 - c_2a_1)x^3y + 4(3b_3d - 3b_1c_2 - b_2c_1)xy^3 + (3b_3^2 - 4c_2b_2)y^4 = 0.$$

Now the condition that this equation,

$$\text{A}x^4 + 4\text{B}x^3y + 6\text{C}x^2y^2 + 4\text{D}xy^3 + \text{E}y^4 = 0,$$

should have equal roots, being

$$(\text{AE} - 4\text{BD} + 3\text{C}^2)^3 = 27(\text{ACE} + 2\text{BCD} - \text{AD}^2 - \text{EB}^2 - \text{C}^3)^2;$$

if we substitute for AB, &c. their values, we shall have a function of the twelfth degree in the coefficients, which is the result of making $c_3 = 0$ in R, and which, therefore, gives us all the terms of R which do not contain c_3. To obtain the other terms, we shall suppose that the equation has been given in the general form, and that the absolute term has been made to vanish by transformation of co-ordinates, then, $x'y'$ being a point on the curve, the new coefficients, expressed in terms of the old, are

$$a_3 = a_3 + a_1x' + a_2y', \quad b_3 = b_3 + b_1x' + b_2y', \quad d = d + a_2x' + b_1y';$$
$$c_1 = c_1 + a_1x'^2 + 2a_2x'y' + b_1y'^2 + 2a_3x' + 2dy';$$
$$c_2 = c_2 + a_2x'^2 + 2b'_1x'y' + b_2y'^2 + 2dx' + 2b_3y';$$

and we can foresee that if we make these substitutions in the partial value of R we shall be able to ged rid of $x'y'$ by the equation

$$a_1x'^3 + 3a_2x'^2y' + 3b_1x'y'^2 + b_2y'^3 + 3a_3x'^2 + 6dx'y' + 3b_3y'^2 + 3c_1x' + 3c_2y' = -c_3,$$

and shall thus obtain the complete value of R.

Now when we calculate the quantity $AE - 4BD + 3C^2$, we find it to be

$$12[d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2) + d(a_1b_3c_2 + b_2c_1a_3) + a_1b_2c_1c_2 - (a_1c_1b_3^2 + a_1b_1c_2^2 + b_2c_2a_3^2 + b_2a_2c_1^2) - (a_2b_1c_1c_2 + b_3c_2a_2a_3 + a_3a_1b_1b_3) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2)].$$

But the reader will perceive that the quantity within the brackets is precisely what S becomes when in it c_3 is made $= 0$.

If we calculate in like manner the value of

$$ACE + 2BCD - AD^2 - EB^2 - C^3,$$

we shall find it to be precisely what T becomes when in it c_3 is made $= 0$. The condition, therefore, that the cubic should have a double point becomes, when the origin is on the curve,

$$64s^3 = t^2,$$

where s and t are what S and T become when in them c_3 is made $= 0$.

204. We have shown in the last Article by what substitutions the complete value of R can be obtained from this partial value, but we may evade this trouble by the following consideration:

We have seen in Art. 198 that either of the conditions $S = 0$, $T = 0$ expresses an absolute peculiarity of the curve, in no way dependent on the axes to which it is referred; and that if $S = 0$ for any one set of axes, the new S must consequently $= 0$ however the axes be transformed. It is plain, then, that however the axes be transformed, the new S and T must be equal to the old S and T multiplied by some numerical factor. And in the case where the transformation is made to parallel axes, since the coefficients a_1, a_2, b_1, b_2 do not change at all, the new S and T must be absolutely equal to the old S and T. It follows, therefore, that the condition that a cubic should have equal roots is in general

$$64S^3 = T^2.$$

I have verified this by the help of the terms of R, which I had

previously obtained by the direct process of elimination, and indeed it was in this manner that I first found the value of T.

205. It has been proved (Art. 158) that the anharmonic ratio is constant of the pencil of four tangents, whose equation is given in Art. 203; and it is desirable to express this ratio in terms of the coefficients of the general equation. Let $\alpha, \beta, \gamma, \delta$ be the four roots of a biquadratic equation, then the anharmonic function of the roots is one of the three quantities,

$$\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}, \quad \frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}, \quad \frac{(\alpha-\delta)(\beta-\gamma)}{(\alpha-\beta)(\gamma-\delta)}.$$

It is easy to see that in order to express these we must first form the equation whose roots are $\alpha\beta+\gamma\delta$, $\alpha\gamma+\beta\delta$, $\alpha\delta+\beta\gamma$. Now if the given equation be

$$Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E = 0,$$

this equation will be (see Lacroix, Complément, § 20)

$$A^3z^3 - 6A^2Cz^2 + A(16BD - 4AE)z - (16AD^2 + 16EB^2 - 24ACE) = 0.$$

We must next form the equation whose roots are the differences of the roots of this latter equation, this will give the equation whose roots are $(\alpha-\beta)(\gamma-\delta)$, $(\alpha-\gamma)(\beta-\delta)$, $(\alpha-\delta)(\beta-\gamma)$, and it will be found to be

$$0 = A^3y^3 - 12A(AE - 4BD + 3C^2)y$$
$$\pm 16\sqrt{\{(AE - 4BD + 3C^2)^3 - 27(ACE + 2BCD - AD^2 - EB^2 - C^3)^2\}},$$

or putting, as before,

$$AE - 4BD + 3C^2 = 12S, \quad 64S^3 - T^2 = R,$$
$$A^3y^3 - 144ASy \pm 16\sqrt{(27R)} = 0.$$

Now the anharmonic functions in question are the ratios of the roots of this equation, and these will not alter if we increase all the roots in the same proportion, by substituting, for example, $Ay = 2t\sqrt{(3S)}$; the equation then becomes

$$t^3 - 12t \pm 2\sqrt{\left(\frac{R}{S^3}\right)} = 0.$$

We see, then, that the anharmonic ratio in question depends solely on the quantity $\frac{R}{S^3}$, or, what is the same thing, on the quan-

tity $\frac{T^2}{S^3}$. This quantity, then, may be taken as what I have elsewhere called the numerical characteristic of the curve.

When $S=0$ the second term of the equation vanishes, and it takes the form $t^3 = \text{const.}$; the roots are therefore of the form $t=c$, $t=\theta c$, $t=\theta^2 c$, where θ is one of the imaginary cube roots of unity, and the ratio in question is expressed by θ.

When $T=0$, $R=64S^3$, and the preceding equation becomes

$$t^3 - 12t + 16 = 0,$$

and has two equal roots. One, therefore, of the ratios in question becomes unity; the anharmonic becomes an ordinary harmonic ratio, and we learn that *when the second Hessian of a cubic is the cubic itself, the four tangents from any point of the curve form a harmonic pencil.*

206. We have already alluded (Art. 165) to the problem to find the curve whose Hessian shall be the given cubic, and we see from Art. 198 that the solution is found by putting $\mu=0$, or

$$108l^3 - 36Slm^2 - Tm^3 = 0,$$

and that there are accordingly three cubics fulfilling the proposed conditions.

We may arrive at the same conclusion by calculating the S and T of the Hessian. This may be done as follows. We have proved

$$H(lU + mH) = (36Sl^2m + 3Tlm^2 + 4S^2m^3)\,U$$
$$+ (108l^3 - 36Slm^2 - Tm^3)\,H.$$

But if we write $H=U_1$, $H_1 = HU_1 = 4S^2U - TU_1$, we may, by the help of this equation, eliminate U, and we have

$$H(lU+mH) = \{432S^2l^3 + 36STl^2m + (3T^2 - 144S^3)lm^2\}\,U_1$$
$$+ (36Sl^2m + 3Tlm^2 + 4S^2m^3)\,H_1.$$

But $lU + mH$ represents the same curve as $(lT + 4S^2m)\,U_1 + lH_1$; if therefore we write $l = m_1$, $4S^2m = l_1 - m_1T$, we find

$$H(l_1U_1 + m_1H_1) = \{3(T^2 - 48S^3)l_1^2m_1$$
$$+ 6T(72S^3 - T^2)\,l_1m_1^2 + 3(T^2 - 48S^3)^2m_1^3\}\,U_1$$
$$+ \{l_1^3 - 3(T^2 - 48S^3)\,l_1m_1^2 - 2T(72S^3 - T^2)\,m_1^3\}\,H_1.$$

Identifying this with the equation

$$H(l_1U_1 + m_1H_1) = (36S_1l_1^2m_1 + 3T_1l_1m_1^2 + 4S_1^2m_1^3)U_1 + (108l_1^3 - 36S_1l_1m_1^2 - T_1m_1^3)H_1,$$

we have

$$S_1 = 9(T^2 - 48S^3), \quad T_1 = 216T(72S^3 - T^2).$$

Now if we write the characteristic of the first curve, $\dfrac{64S^3}{T^2} = a$ that of the second curve will be

$$\frac{64S_1^3}{T_1^2} = 1 + \frac{27a^2(1-a)}{(8-9a)^2},$$

which gives a cubic equation to determine a when the characteristic of the second curve is known.

It may be proved in like manner that

$$S(lU + 6mH) = Sl^4 + Tl^3m + 24S^2l^2m^2 + 4STlm^3 + (T^2 - 48S^3)m^4,$$

$$T(lU + 6mH) = Tl^6 + 96S^2l^5m + 60STl^4m^2 + 20T^2l^3m^3 + 240TS^2l^2m^4 + 48(ST^2 + 96S^4)lm^5 + 8(72S^3T - T^3).$$

Hence through the nine points of intersection of U and H can be drawn four systems of right lines, four cubics fulfilling the condition $S = 0$, and six *harmonic* cubics, as from the property of Art. 205 I call the cubics which fulfil the condition $T = 0$.

If we form $R(lU + 6mH)$ by calculating $64S^3 - T^2$, the result will be found

$$R(l^4 - 24Sl^2m^2 - 8Tlm^3 - 48S^2m^4)^3,$$

as we might have foreseen, since, if the curve U have not a double point, the only cubics with double points which can be drawn through the points of inflexion are the four systems of right lines.

207. When a cubic has a cusp three of the tangents which can be drawn from any point of it coincide with the line joining that point to the cusp; hence the conditions for a cusp are, that the biquadratic in Art. 203 should have three roots equal; and these conditions are $S = 0$, $T = 0$. If we wish to know how many cubics of the third class can be drawn through seven given points, the equation of a system of cubics through seven given points is $lA + mB + nC = 0$, and we must determine $l : m$, $l : n$, so that the coefficients shall satisfy the equations $S = 0$, $T = 0$. But since one of these equations is of the fourth, and the other of the sixth degree, in the coefficients, it appears that the problem admits of twenty-four solutions.

208. It seems desirable to add something as to the method of obtaining the two conditions which must be fulfilled in order that an equation of the third degree should be resolvable into the product of one of the first degree by one of the second. Mr. Cayley has remarked (Cambridge and Dublin Journal, i. 103), that if $U = PV$; then $H(U)$ is of the form $P(\rho V + \sigma P^2)$. If we have

$$P = lx + my + nz,$$

$$V = Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2,$$

then $\rho = (CF - E^2)l^2 + (AF - D^2)m^2 + (AC - B^2)n^2$
$$+ 2(BD - AE)mn + 2(BE - CD)nl + 2(DE - BF)lm;$$

$$\sigma = 4(AE^2 + CD^2 + FB^2 - ACF - 2BDE);$$

or $\rho = 0$ is the condition that the right line should touch the conic, and $\sigma = 0$ the condition that the conic should break up into two right lines. It will be found also that in this case $S(U) = L^2$, $T(U) = 8L^3$, so that $H + U\sqrt{(S)}$ will be a perfect cube. It follows also from the equation

$$H = \rho U + \sigma P^3,$$

$$\frac{dH}{dx} = \rho\frac{dU}{dx} + 3l\sigma P^2,$$

$$\frac{dH}{dy} = \rho\frac{dU}{dy} + 3m\sigma P^2,$$

$$m\frac{dH}{dx} - l\frac{dH}{dy} - m\rho\frac{dU}{dx} + l\rho\frac{dU}{dy} = 0;$$

and hence

$$\left\| \begin{matrix} a_1, & a_2, & b_1, & a_3, & d, & c_1, \\ a_2, & b_1, & b_2, & d, & b_3, & c_2, \\ \mathrm{a}_1, & \mathrm{a}_2, & \mathrm{b}_1, & \mathrm{a}_3, & \mathrm{d}, & \mathrm{c}_1, \\ \mathrm{a}_2, & \mathrm{b}_1, & \mathrm{b}_2, & \mathrm{d}, & \mathrm{b}_3, & \mathrm{c}_2, \end{matrix} \right\| = 0,$$

where the equation denotes the result of putting $= 0$ *all* the fifteen determinants formed by taking any four vertical columns. So, in like manner, we may obtain two other sets of determinants by substituting $\frac{dU}{dz}, \frac{dH}{dz}$ for $\frac{dU}{dx}, \frac{dH}{dx}$, or $\frac{dU}{dy}, \frac{dH}{dy}$. It would be interesting to verify that these are all equivalent only to two independent conditions, but I am unable to do this, having, until the occasion of writing this, omitted to consider the case where a cubic

breaks up into a right line and a conic. And for the same reason I cannot be positive that these conditions, which are of the eighth degree in the coefficients, are the simplest which may be obtained. I had supposed that it would be possible to obtain conditions of the sixth degree in the coefficients by the method of Art. 203, that is, by forming the conditions that the biquadratic there given should have two distinct pairs of equal roots; but unfortunately this is not necessarily the case when the equation breaks up into factors; it will be true when the origin is a point on the conic, but not so, if the origin be anywhere on the right line.

I may add, that if c_1, c_2, c_3 be all $= 0$, the origin is a double point, and the condition that the cubic should have a second double point is of the fifth degree in the coefficients, and enters as a factor into the coefficients in R of the lowest powers of c_1, c_2, c_3.*

CHAPTER IV.

CURVES OF THE FOURTH DEGREE.

209. THERE is no difficulty in seeing how curves of the fourth degree are to be classified. If they have no multiple point they will be of the twelfth class; but they may have a triple point, in which case they will be of the sixth, fifth, or fourth class, according as the tangents at the triple point are either all distinct, or two coincident, or all three coincident; or else the curve may have one, two, or three double points, any or all of which may be cusps; and the class of the curve will accordingly vary from the third to the tenth inclusive. Two of the double points may unite

* This section is derived from a memoir of M. Aronhold's, published in Crelle's Journal, vol. xxxix. p. 140. M. Aronhold's paper being preliminary to a promised larger one, contains chiefly theorems without proof; and I must confess that the proofs and developments which I have accordingly been obliged to supply, savour rather of brute force than of science. In everything relating to the manipulation of determinants, mathematics are making rapid progress, and no doubt there will soon be published easier methods of arriving at the results here given.

into such a point as that noticed, p. 29, the tangent at which meets the curve in four consecutive points.

Or, if we choose to classify these curves according to the number of their infinite branches, we can see that the line at infinity meets the curve either in four real points, four imaginary, or in two real and two imaginary; the line at infinity may be a tangent or a double tangent, or may touch at a point of inflexion or osculation. Or again, there might be one or two multiple points at infinity. There would be no difficulty in making a table of all the varieties which might thus arise, but it is useless to take up space with it here, as we are not in a position to give an analysis of the figures of all possible curves of the fourth degree.

Comparatively little attention having been given by geometers to curves of the fourth order, we shall give in this Chapter the few general properties which have been obtained, and shall add some notices of those particular curves which have received distinctive names.

210. We have already given (p. 71) the method of finding the curve (H) of the sixth degree, which determines the twenty-four points of inflexion of a curve of the fourth degree, and have given (p. 89) the equation of the curve which passes through the points of contact of its twenty-eight double tangents; we have given (p. 101) the general equation of its reciprocal, which there led us to the property that the tangents at the twenty-four points of inflexion touch a curve of the fourth class. We proceed now to give some theorems relative to the double tangents of the curve.

211. Let x, y be any two of the double tangents, then the equation of the curve must be of the form

$$xyU = V^2,$$

where U and V are functions of the second degree; but this may also be written in the form

$$xy(U + 2\lambda V + \lambda^2 xy) = (V + \lambda xy)^2.$$

Now we can determine λ so that $U + 2\lambda V + \lambda^2 xy$ shall represent two right lines, and it will be seen that this will give an equation of the fifth degree for λ; hence in five different ways the given equation can be brought to the form

$$wxyz = V^2;$$

and we see that *through the four points of contact of any two double tangents can be described five conic sections, each of which also passes through the four points of contact of two other double tangents.*

We can take $\frac{28 \cdot 27}{2}$ different pairs of double tangents; each of these pairs gives rise to five conic sections, but each conic section may arise from any one of six different pairs of double tangents: hence *there may be described* 315 *conics, each passing through eight of the points of contact of double tangents to a curve of the fourth degree.**

212. We shall form a scheme of these 315 conics: we denote the first twenty-six double tangents by the letters of the alphabet, adding the symbols ϕ, ψ for the twenty-seventh and twenty-eighth; and we denote by $abcd$ the conic passing through the eight points of contact of the double tangents a, b, c, d. We must first prove that if $abcd$, $abef$, be two of the conics, then $cdef$ will be another. For, by the last Article we shall have

$$ef = cd + 2\lambda V + \lambda^2 ab;$$

and, substituting the value of V derived from this equation, the equation of the curve becomes

$$\lambda^4 a^2 b^2 + c^2 d^2 + e^2 f^2 - 2\lambda^2 abcd - 2cdef - 2\lambda^2 efab = 0,$$

or $$4cdef = (cd + ef - \lambda^2 ab)^2. \quad \text{Q. E. D.}$$

Again, if $abcd$, $abef$ be two of the conics, $aceg$ cannot be another unless the curve have a double point. For we have seen that the conic through the points of contact of $abcd$ is $\lambda ab + \frac{1}{\lambda}(cd - ef)$; but if $aceg$ be another conic, then the conic through $abcd$ must likewise be $\mu ac + \frac{1}{\mu}(bd - eg)$; hence

$$\lambda ab + \frac{1}{\lambda}(cd - ef) = \mu ac + \frac{1}{\mu}(bd - eg),$$

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* M. Plücker first noticed the possibility of bringing an equation of the fourth degree to the form $wxyz = V^2$; but he hastily inferred that the six points of contact of *any* three double tangents lie on a conic, and thence drew an erroneous conclusion as to the total number of conics passing through eight points of contact of double tangents. (See Theorie der Algebraischen Curven, p. 246).

$$\left(a - \frac{1}{\lambda\mu}d\right)(\lambda b - \mu c) = e\left(\frac{1}{\lambda}f - \frac{1}{\mu}g\right);$$

e, therefore, is passes either through ad or bc, and it is plain then that one or other of these points must be a double point on the curve
$$4\lambda^2 abcd = (\lambda^2 ab + cd - ef)^2.$$
It is in like manner proved that if $abef$, $acmn$ be two conics, the diagonals of the quadrilateral $efmn$ pass, one through ad, and the other through bc.

213. Attending to these two remarks, we form the following scheme, which has nothing arbitrary in it but the notation:

$abcd,\ abef,\ abgh,\ abij,\ abkl,\ acmn,\ acop,\ acqr,\ acst,\ aduv,$
$adwx,\ adyz,\ ad\phi\psi,\ aemu,\ aeow,\ aeqy,\ aes\phi,\ afnv,\ afpx,\ afrz,$
$aft\psi,\ agmx,\ agov,\ agr\phi,\ agty,\ ahnw,\ ahpu,\ ahq\psi,\ ahsz,\ aimz,$
$aiqv,\ aip\phi,\ aitw,\ ajny,\ ajru,\ ajo\psi,\ ajsx,\ akm\psi,\ aksv,\ akpy,$
$akrw,\ aln\phi,\ altu,\ aloz,\ alqx,\ bcuv,\ bcwx,\ bcyz,\ bc\phi\psi,\ bdmn,$
$bdop,\ bdqr,\ bdst,\ benv,\ bepx,\ berz,\ bet\psi,\ bfmu,\ bfow,\ bfqy,$
$bfs\phi,\ bgnw,\ bgpu,\ bgq\psi,\ bgsz,\ bhmx,\ bhov,\ bhr\phi,\ bhty,\ biny,$
$biru,\ bio\psi,\ bisx,\ bjmz,\ bjqv,\ bjp\phi,\ bjtw,\ bkn\phi,\ bktu,\ bkoz,$
$bkqx,\ blm\psi,\ blsv,\ blpy,\ blrw,\ cdef,\ cdgh,\ cdij,\ cdkl,\ cenu,$
$cepw,\ cery,\ cet\phi,\ cfmv,\ cfox,\ cfqz,\ cfs\psi,\ cgnx,\ cgpv,\ cgq\phi,$
$cgsy,\ chmw,\ chou,\ chr\psi,\ chtz,\ cinz,\ cio\phi,\ cirv,\ cisw,\ cjmy,$
$cjp\psi,\ cjqu,\ cjtx,\ ckn\psi,\ ckoy,\ ckqw,\ cktv,\ clm\phi,\ clsu,\ clpz,$
$clrx,\ demv,\ deox,\ deqz,\ des\psi,\ dfnu,\ dfpw,\ dfry,\ dft\phi,\ dgmw,$
$dgou,\ dgr\psi,\ dgtz,\ dhnx,\ dhpv,\ dhq\phi,\ dhsy,\ dimy,\ dip\psi,\ diqu,$
$ditx,\ djnz,\ djo\phi,\ djrv,\ djsw,\ dkm\phi,\ dksu,\ dkpz,\ dkrx,\ dln\psi,$
$dloy,\ dlqw,\ dltv,\ efgh,\ efij,\ efkl,\ egqt,\ egrs,\ egux,\ egvw,$
$ehmp,\ ehno,\ ehy\psi,\ ehz\phi,\ eiot,\ eips,\ eiuz,\ eivy,\ ejmr,\ ejnq,$
$ejx\phi,\ ejw\psi,\ ekor,\ ekpq,\ eku\psi,\ ekv\phi,\ elmt,\ elns,\ elwz,\ elxy,$
$fgmp,\ fgno,\ fgy\psi,\ fgz\phi,\ fhqt,\ fhrs,\ fhux,\ fhvw,\ fimr,\ finq,$
$fix\phi,\ fiw\psi,\ fjot,\ fjps,\ fjuz,\ fjvy,\ fkmt,\ fkns,\ fkwz,\ fkxy,$
$flor,\ flpq,\ flu\psi,\ flv\phi,\ ghij,\ ghkl,\ gioq,\ gipr,\ gixz,\ giwy,$
$gjms,\ gjnt,\ gju\phi,\ gjv\psi,\ gkos,\ gkpt,\ gkw\phi,\ gkx\psi,\ glmq,\ glnr,$
$gluy,\ glvz,\ hims,\ hint,\ hiu\phi,\ hiv\psi,\ hjoq,\ hjpr,\ hjxz,\ hjwy,$
$hkmq,\ hknr,\ hkuy,\ hkvz,\ hlos,\ hlpt,\ hlw\phi,\ hlx\psi,\ ijkl,\ ikqs,$
$ikrt,\ iky\phi,\ ikz\psi,\ ilmo,\ ilnp,\ iluw,\ ilvx,\ jkmo,\ jknp,\ jkuw,$
$jkvx,\ jlqs,\ jlrt,\ jly\phi,\ jlz\psi,\ mnop,\ mnqr,\ mnst,\ mouw,\ movx,$
$mp\phi z,\ mpy\psi,\ mqvz,\ mquy,\ mrx\phi,\ mrw\psi,\ msu\phi,\ msv\psi,\ mtwz,\ mtxy,$

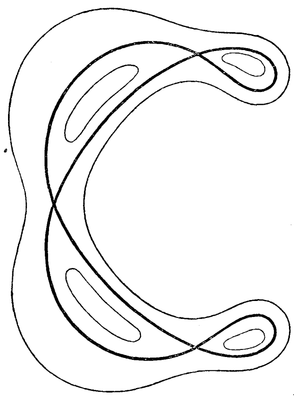
noyψ, *nozφ*, *npuw*, *npvx*, *nqwψ*, *nqxφ*, *nruy*, *nrvz*, *nswz*, *nsxy*, *ntuφ*, *ntvψ*, *opqr*, *opst*, *oqwy*, *oqxz*, *oruψ*, *orvφ*, *oswφ*, *osxψ*, *otuz*, *otvy*, *pquψ*, *pqvφ*, *prwy*, *prxz*, *psuz*, *psvy*, *ptwφ*, *ptxψ*, *qrst*, *qszψ*, *qsyφ*, *qtux*, *qtvw*, *rsux*, *rsvw*, *rtyφ*, *rtzψ*, *uvwx*, *uvyz*, *uvφψ*, *wxyz*, *wxφψ*, *yzφψ*.

These conics may be divided in more ways than I have had patience to count, into sets of seven passing through all the points of contact. For example, *abcd*, *efgh*, *ijkl*, *mnop*, *qrst*, *uvwx*, *yzφψ*. And for the pair *efgh*, *ijkl* we might substitute *efij*, *ghkl*, or for the pair *efgh*, *mnop* we might substitute *ehmp*, *fgno*, &c.*

214. M. Plücker has given the following example (Theorie, p. 247) to show that the twenty-eight double tangents may be all real. He takes an equation of a curve having three double points, for example:

$$\Omega = (y^2 - x^2)(x-1)(x-\tfrac{3}{2}) - 2\{y^2 + x(x-2)\}^2 = 0,$$

which is represented by the dark curve in the figure. Now the equation



$$\Omega \pm k = 0$$

denotes a curve not meeting Ω in any finite point, which deviates the less from the form of the curve Ω the less we suppose k; and which, according to the sign we give k, is either altogether within or altogether without the curve Ω. In the case when it is altogether within Ω, it is easy to see that the curve consists of four meniscus-shaped ovals, each of which has its own double tangent, and besides, any of the six pairs of ovals has four common tangents.

* My attention was directed to this question by the following note of M. Hesse's (Crelle, vol. 40, p. 260). He says that he was led to foresee the possibility of forming an equation of the fourteenth degree, satisfied for the fifty-six points of contact, from his knowledge of the fact that these points lie on a system of seven conics. "Ich kann 7 Kegelschnitte angeben, welche durch sämmtliche Berührungspuncte hindurchgehen, nicht auf die Weise, wie der unrichtige *Plücker*sche Satz über die Kegelschitte, welche die Curve in den Berührungspuncten schneiden sollen, vermuthen liesse, sondern auf eine

This example is also instructive, as suggesting how curves of the fourth degree and twelfth class are to be subdivided. We saw (Chap. III. Sect. III.) that curves of the third degree were either altogether continuous, or bipartite, that is to say, consisting of a continuous infinite part, together with an oval or the projection of an oval. Here we see that curves of the fourth degree may have four (and I suppose any smaller number of) separate parts, and they may be subdivided accordingly.

215. From the consideration that any curve which meets an oval once must meet it again, we can obtain a limit to the number of ovals which a curve of any degree can possess; just as (p. 31) we obtained a limit to the number of its double points. A curve of the third degree cannot have two ovals, for then the line joining a point on one to a point on the other would meet the curve in four points. A curve of the fourth degree, if it have four ovals, can have no other point; for then the conic through this point and one on each of the four ovals would meet the curve in nine points. So in like manner a curve of the fifth degree cannot have more than six ovals, though it may have an infinite branch besides. And when a curve of the fifth degree has so many ovals, each must be altogether external to all the rest. For if there be any point within both the first and second, the line joining this to a point within the third would meet the curve in six points. Every pair of ovals must then have four common tangents, and we can thus account for 60 of the 120 double tangents of a curve of the fifth degree.

216. We have nothing to add as to the general properties of curves of the fourth degree and twelfth class. Of curves with a double point, the most interesting case is when that point is a point of inflexion on both the branches which pass through it; that is to say, when both its tangents meet the curve in four consecutive points. If, in general, we look for the locus of harmonic means on radii drawn through a double point on the curve

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ganz andere Art, die ich wegen ihrer Weitlaüftigkeit hier nicht angeben kann." I do not know whether M. Hesse's conics are the same as mine, for, as he speaks of his conics as quite different from M. Plücker's, perhaps in M. Hesse's method the eight points which lie on a conic may belong to eight different double tangents.

$u_4 + u_3 + u_2 = 0$, we find, as at p. 139, $u_3 + 2u_2 = 0$; when, therefore, u_2 is a factor in u_3, the locus becomes a right line, and the double point enjoys properties such as those possessed by the points of inflexion of cubics. The points of contact of tangents from it lie in a right line, and the curve may be projected so as that this point should become a centre, or else so that all chords parallel to a given line should be bisected by a fixed diameter. In the latter case the form of the equation is, in general,

$$y^2 (x - a) (x - b) = \pm \mathrm{A} (x - c) (x - d) (x - e) (x - f),$$

and there is no difficulty in discussing, as in Sect. III. of the last Chapter, the different possible forms of curves included in this equation, and the different possible forms of their projections.

Curves of the fourth degree with two double points may all be projected, by a real or imaginary transformation, to the class discussed, p. 125, where the double points are the two imaginary circular points at infinity. From each of the double points may be drawn a pencil of four tangents to the curve, and it follows from the property proved, p. 126, that the anharmonic functions of the two pencils are equal.

217. Curves of the fourth degree with three real double points are all included in the equation

$$x^2y^2 + y^2z^2 + z^2x^2 + 2xyz (\mathrm{A}x + \mathrm{B}y + \mathrm{C}z) = 0;$$

the pairs of tangents at the double points being (Art. 42)

$$y^2 + z^2 + 2\mathrm{A}yz = 0, \quad z^2 + x^2 + 2\mathrm{B}zx = 0, \quad x^2 + y^2 + 2\mathrm{C}xy = 0;$$

and the form of these equations shows (*Conics*, p. 244) that these six tangents all touch the same conic. When the curve has three double points it has, by the general theory (see p. 91), six points of inflexion, and four double tangents; and if the double tangents be represented by t, u, v, w, the equation of the curve may be written in the form

$$t^{\frac{1}{2}} + u^{\frac{1}{2}} + v^{\frac{1}{2}} + w^{\frac{1}{2}} = 0.$$

For, throwing the equation into the form

$$(t^2 + u^2 + v^2 + w^2 - 2tu - 2uv - 2vw - 2wt - 2tv - 2uw)^2 = 64tuvw,$$

it appears that t, u, v, w are double tangents to the curve, and it can immediately be verified that $(t - u, v - w)$, $(t - v, u - w)$,

$(t - w, u - v)$, are double points, and the equation containing implicitly eleven $(= 14 - 3)$ constants is sufficiently general to represent all curves of the fourth degree with three double points. The equation shows, then, that the eight points of contact of double tangents lie on the same conic.

On the subject of the pairs of tangents which can be drawn from the double points to the curve, see a paper by Mr. Cayley (Cambridge and Dublin Journal, v. 148); when the points of contact of these lie three by three on two right lines, the equation can be reduced to the form

$$\{au\,(v + w - u)\}^{\frac{1}{2}} + \{bv\,(w + u - v)\}^{\frac{1}{2}} + \{cw\,(u + v - w)\}^{\frac{1}{2}} = 0.$$

When the curve has three cusps, then, in the first equation of this Article, $A = B = C = -1$, and the equation becomes

$$x^{-\frac{1}{2}} + y^{-\frac{1}{2}} + z^{-\frac{1}{2}} = 0,$$

the reciprocal of the class discussed, Art. 184. The equation shows that the three tangents at the cusps meet in a point, as we can at once perceive by reciprocation. When the curve has two cusps and a node, the line joining the two points of inflexion, the line joining the two cusps, and the double tangent, all pass through the same point.

Of curves with a triple point the most interesting case is when the three tangents coincide, and the equation takes the form $x^3y = z^4$; the curve has then, beside the triple point, no other singular point, save a point of undulation, and its reciprocal is a curve of similar nature.

The properties of this curve may be discussed as in Sect. VI. of the last Chapter.

218. It remains to mention some of the most remarkable species of curves of the fourth degree.

We have already mentioned (p. 122) some of the properties of the *ovals of Des Cartes*. These are, in general, curves of the sixth class, having the two imaginary circular points for cusps. They consist of a pair of conjugate ovals, the two ovals answering to the double sign in the equation $\rho \pm m\rho' = c$. It can easily be seen geometrically that a Cartesian oval may be considered as the locus of the vertex of a triangle whose base angles move on two

given circles, while the two sides pass through the centres of the circles, and the base passes through a fixed point on the line joining them. If the axis meet the curve in the four points A, B, C, D, the three foci will be the points where the axis is cut, so that AF.FB = CF.FD, or so that AF.FC = BF.FD, or so that AF.FD = BF.FC.*

If any line meet a Cartesian oval in four points, the sum of their four distances from any focus is constant: for the polar equation of the curve is of the form

$$\rho^2 - 2(a + b\cos\omega)\,\rho + c^2 = 0;$$

and if we eliminate ω between this and the equation of an arbitrary right line, we get a biquadratic for ρ, of which $-4a$ is the second term.

When in the preceding equation $c = 0$, the equation becomes $\rho = a + b\cos\omega$, the origin is a double point, and the curve is of the fourth class, and is the "*limaçon de Pascal.*" It may evidently be generated by taking a constant length on the radii vectores to a circle from a point on it.

When $b = a$, the equation becomes of the form $\rho^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{1}{2}\omega$; the origin is a cusp, and the *curve* is the *cardioide,* a curve generated by adding or subtracting a portion equal to the diameter on the radii vectores to a circle from a point on it. We shall mention some other properties of this curve in the next Chapter.

219. We have also mentioned already (p. 127) some of the properties of the *ovals of Cassini*, or the locus of the vertex of a triangle, when the base and rectangle under sides is given. The origin being the middle point of the base, the polar equation is

$$\rho^4 - 2c^2\rho^2\cos 2\omega + c^4 = m^4.$$

The circular points at infinity are double points of the nature of those discussed, Art. 216, and the curve is of the eighth class.

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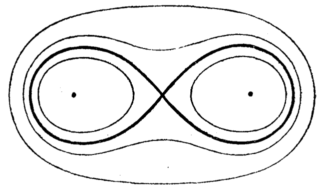
* On the subject of Cartesian ovals, see Chasles, Aperçu Historique, p. 350; Quetelet, Nouveaux Mémoires de Bruxelles, tom. v. M. Chasles only noticed that the circular points were double points, and accordingly speaks of these curves as of the eighth class, and, strangely enough, no correction of this oversight appears to have been published until Mr. Cayley's paper, Liouville, vol. xv. p. 354.

If $c^4 = m^4$, the equation becomes $\rho^2 = 2c^2 \cos 2\omega$, the origin is a double point, and the curve is the *lemniscata of Bernouilli*, consisting of two ovals joined into a kind of figure of 8. This is represented by the dark curve on the figure; when m is less than c, Cassini's ovals consist of two conjugate ovals within the parts of this figure; when m is greater than c, of one continuous oval outside it.



This lemniscata is the locus of the foot of the perpendicular from the centre on the tangent to an equilateral hyperbola. The locus, in general, of the foot of the perpendicular from the centre of any conic on the tangent is obviously

$$\rho^2 = a^2 \cos^2\omega + b^2 \sin^2\omega,$$

a curve having the origin for a double point, and the two circular points at infinity for ordinary double points.

As a generalization of the ovals of Cassini, we might seek the locus of a point, the product of whose distances from m given points shall be constant; and when the m points are the vertices of a regular polygon, the polar equation referred to the centre of the polygon is easily seen to be

$$\rho^{2m} - 2a\rho^m . \cos m\omega + a^{2m} = b^{2m},$$

becoming, when $a = b$, $\rho^m = 2a^m \cos m\omega$.

220. Another remarkable curve of the fourth degree is the *conchoid of Nicomedes,* invented by that geometer for the solution of the problem of finding two mean proportionals. It is generated by taking on the radius vector from a fixed point O to a fixed line MN, a portion RP of given length on either side of the right line. The curve admits of an obvious mechanical description, since we have only to imagine OR to be a grooved rule capable of turning round O, while the point R is made to slide along another grooved rule MN; then a pencil placed at any fixed point P of the rule OR will describe the conchoid. The polar equation of the curve is immediately found, and is, if OA $= p$, RP $= m$,

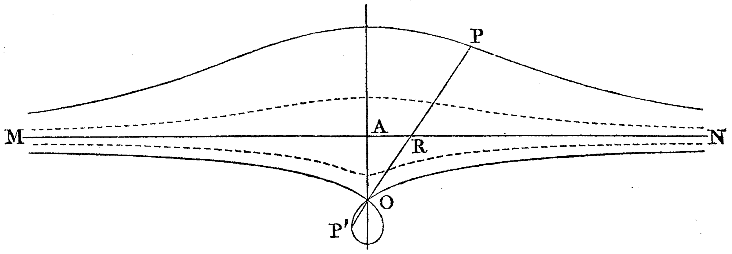
$$(\rho \pm m) \cos\omega = p,$$

or, in rectangular co-ordinates,

$$m^2x^2 = (p - x)^2 (x^2 + y)^2.$$

The form of the equation shows that the line MN $(p - x)$ touches at a double point at infinity, and there meets the curve in four consecutive points; and since at this point MN is the only tangent, it is a point resulting from the union of two double points. The point O is also a double point, the tangents at which are given by the equation

$$p^2y^2 + (p^2 - m^2) x^2 = 0.$$



It will therefore be a node, conjugate point, or cusp, according as m is greater, less than, or equal to p. The dark figure represents the case when the curve has a node; the dotted figure denotes the case when p is greater than m. The curve is of the fifth class when $p = m$, otherwise of the sixth.

CHAPTER V.

TRANSCENDENTAL CURVES.

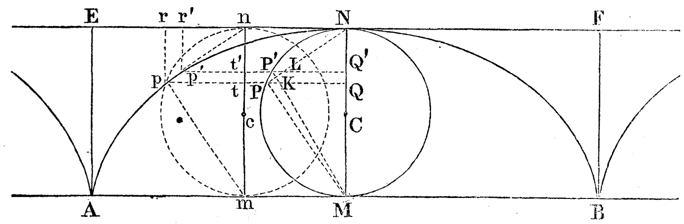
221. We have hitherto exclusively discussed equations reducible to a finite number of terms involving positive integer powers of x and y; it remains to mention something of the properties of curves represented by transcendental equations. Since these involve functions only expressible by an infinite series of algebraical terms, all transcendental curves may be considered as curves of infinite degree; they may be cut by any right line in an infinity of points, and must have an infinity of multiple points

and multiple tangents. There is then no room for a general theory of the singularities of these curves, and it is only necessary to mention the names and principal properties of some of the most remarkable of them. We may notice, in passing, a class of equations, called by Leibnitz *interscendental*, or which involve the variables with exponents not commensurable with any rational number; for example, $y = x^{\sqrt{2}}$. Here, as we successively substitute for $\sqrt{2}$ the series of rational fractions which approximately express the value of the radical, we shall find a series of algebraic curves of constantly increasing degree, more and more nearly resembling the figure of the required curve, but not accurately expressing it as long as the degree of the curve is finite. We pass on to the *cycloid*, which holds the first place among transcendental curves, both for historical interest and for the variety of its physical applications. This curve is generated by the motion of a point on the circumference of a circle which rolls along a right line. Let A be the point where the motion commences; then, in any position of the generating circle, if p be the generating point, we must have the arc $pm = \text{A}m$, and denoting the angle pcm by ϕ, and cm, the radius of the circle, by a, we shall have

$$y = a(1 - \cos\phi), \quad x = a(\phi - \sin\phi);$$

whence, eliminating, we shall have the equation of the curve,

$$a - y = a\cos\left\{\frac{x + \sqrt{(2ay - y^2)}}{a}\right\}.$$



It is, however, generally more convenient to retain ϕ, and to consider the curve as represented by the two equations given above. It is easily seen that the form of the curve is that represented in the figure; and since the circle may roll on indefinitely in either direction, that the curve consists of an infinity of similar portions,

and that there is a cusp at the point of union of any two such portions.

Let mpn be the position of the generating circle corresponding to the highest point of the cycloid, then since Am = arc pm, AM = MPN, we have Mm = pP = arc PN; or the curve is generated by producing the ordinates of a circle until the produced part be equal to the corresponding arc, measured from the extremity of the diameter. Denoting the angle PCN by θ, the curve referred to the axes AM, MN is represented by the equations

$$y = a(1 + \cos\theta) \quad x = a(\theta + \sin\theta).$$

222. We can readily see how to draw a tangent to the curve, for at any instant of the motion of the generating circle, m (its lowest point) is at rest, and the motion of every point of the circle is for the moment the same as if it described a circle about m; hence the normal to the locus of p must pass through m, and its tangent must always be parallel to NP. The same thing appears analytically for $\dfrac{dy}{dx} = \dfrac{\sin\phi}{1 - \cos\phi} = \cot\tfrac{1}{2}\phi$; the tangent therefore makes with the axis of x an angle the complement of CNP, which is $\frac{1}{2}\phi$.

It is so easy to give geometrical proofs of some of the principal properties of the cycloid that we add them here. *The area of the curve is three times the area of the generating circle.* For the element of the external area ($pp'rr' = pp'tt'$ = PP′QQ′) is equal to the element of the area of the circle; the whole external area, therefore, AENFB, is equal to the area of the circle; and therefore the internal area, ANB, is three times the area of the circle.

The arc Np *of the cycloid is double* NP *the chord of the circle.*

For it is easy to see that the triangle PP′L is isosceles, and therefore that if a perpendicular, MK, be let fall on the base, PL the increment of the arc of the cycloid, is double PK the increment of the chord of the circle.

Hence if s denote the arc of the cycloid, b the diameter of the generating circle, x the abscissa NQ from the vertex; then the equation of the curve is $s^2 = 4bx$, a form useful in Mechanics.

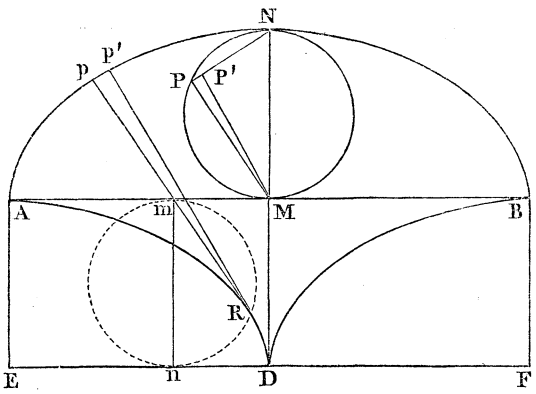
The radius of curvature is double the normal.

For the triangle formed by two consecutive normals has its

sides parallel to those of the triangle MPP′, but the base of the first triangle is equal to PL, and, as we have just proved, is double PK, the base of the second; hence the radius of curvature is double MP.

The evolute of the cycloid is an equal cycloid.

For if we suppose a circle touching the base at m, and passing through R, the centre of curvature, it is equal to the generating circle, and the arc, nR, is equal to NP = nD; hence the locus of R is the cycloid described by the circle mRn rolling on the base EF.*



* The following sketch of the history of the cycloid is abridged from Montucla's History of Mathematics. The first notion of the cycloid occurs in the writings of the Cardinal De Cusa, who, in his attempts to obtain the quadrature of the circle, was led to imagine a circle rolling along a right line until it had traversed on that line a portion equal to its circumference. The properties, however, of the curve generated by a point on the rolling circle do not appear to have formed a subject of his inquiries. More than a century afterwards it occurred to Mersenne, a French friar of the order of Minims, to consider the nature of the path described by a point on the rim of a wheel, and he obtained the more obvious properties of the curve, such as that the length of the base is equal to the circumference of the generating circle, and that the curve is generated by producing the ordinates of a circle until the produced part be equal to the arc. Mersenne also attempted to obtain the quadrature of the curve, and, failing himself, proposed the problem to several other mathematicians. Meanwhile, on the other side of the Alps, the properties of the cycloid had engaged the attention of Galileo, who, in a letter to Torricelli in 1639, in which he speaks of the curve as a graceful form for the arches of a bridge, claims an acquaintance with the curve of forty years' standing. Galileo also endeavoured to compare the area of the cycloid with that of its generating circle, and, having exhausted his geometry on the

We might also seek the locus of any point in the plane of the generating circle carried round with it; when the point is inside the circle the locus is called the prolate cycloid; when it is outside it is called the curtate cycloid: these loci are by some called trochoids. There is no difficulty in calculating their equation or in ascertaining their figures, but it does not seem worth while to

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problem in vain, had recourse to the expedient of weighing the one against the other. Finding, on repeated experiments, their ratio to be nearly, but not exactly, 3 to 1, he concluded that it could not be expressed by rational numbers. Mr. J. S. Mill (in a passage in his Logic, on which I cannot lay my hand) erroneously represents Galileo's experiments to have been successful, and, if I remember right, looks on the proceeding with more complacency than I can regard it. Though it be true that all geometry ultimately reposes on facts made known by the senses, still they are facts which form a necessary part of every one's observation, so that the reader of a mathematical demonstration feels that he is not called on to receive anything on the evidence of the testimony of another, but that he is himself able to bear independent testimony to the truth of what he reads; but were the Galilean method of quadratures to become general, a mathematical treatise would be a list of isolated experiments, the truth of which must be taken on trust by any reader not disposed to verify them all for himself. And even were he to do so, all that he could be sure of (in the present instance, for example) would be, that the area of the cycloid differed from three times that of the circle by an amount less than a certain quantity depending on the possible errors of the experiment. And when the cycloid and the three circles had been found to balance each other in one pair of scales, it would still be wholly impossible to predict whether they would continue to do so, if tried by a more delicate instrument. Galileo's experiment has been defended by Groningius, from the example of Archimedes, who gave a quadrature of the parabola on mechanical principles; but in this demonstration Archimedes only made use of those abstract principles of Statics, the truth of which must be recognised by all. He did not obtain by experiment a new physical fact, but derived by reasoning a new consequence from facts known to every one already.

To return to the history of the cycloid: Mersenne proposed the problem of its quadrature to Roberval in the year 1628. The latter was at that time unequal to the task of its solution, but having devoted himself to the study of the geometry of the Greeks, and of Archimedes in particular, he resumed the problem with success in 1634. He appears himself to have regarded his achievement with no small triumph, though his six years' gestation did not escape the sneers of his contemporaries. When Des Cartes was informed by Mersenne of Roberval's discovery, he wrote back in rather a contemptuous tone, speaking of the theorem as a very pretty one, which he had not noticed before, but still which could cause no difficulty to any moderately skilful geometer, and enclosing at the same time a solution of the problem. Roberval, however, not being convinced that Des Cartes would have found the question so easy, had he not had the advantage of being informed beforehand what its solution was, Des Cartes challenged him to find the method of drawing a tangent to the curve, he having himself, in the mean time, discovered the very sim-

devote any space to them here. The method of drawing tangents given for the cycloid, applies equally to these curves. These curves may (as the reader can easily see) be generated by a point on the circumference of a circle rolling so that the arc pm shall be *in a constant ratio* to the line Am.

223. When the properties of the cycloid had been investigated, it was a natural extension to discuss the curve traced by a

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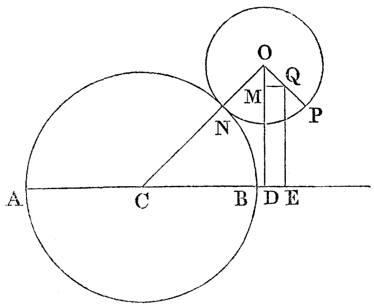
ple solution which we have adopted in the text. The problem baffled Roberval, but was solved by the celebrated mathematician Fermat, who was also included in Des Cartes' challenge. The same problems were afterwards independently solved by the Italian geometers; that of quadrature by Torricelli, and of tangents by Viviani. Some other properties of the cycloid were discovered in the following years, but it was in 1658 that the curve again engrossed the attention of the mathematical world. The wonderful genius of Pascal had been early lost to geometry. In his twenty-fourth year, when he had already given proof of abilities sufficient to place him in the first rank of mathematicians, he abandoned these unprofitable speculations, believing that preparation for eternity ought to engross all the thoughts of an immortal being. For twelve years he maintained his stern resolution, until attacked by the malady which ultimately proved fatal to him. This disease commenced with a violent toothache, which totally deprived him of sleep. One night, as he was tossing in agonies of pain, his thoughts reverted to the subject of his early studies. A few sleepless nights sufficed to put him in possession of a number of curious properties of the cycloid, the quadrature and centre of gravity of any segment, the volume and surface of the solids formed by its revolution round either the base or the axis. Pascal was disposed to consign his discoveries to oblivion, but he yielded to the remonstrances of his friends, who represented to him how much he might advance the cause of religion were he to exhibit in his own person the union of the most sincere faith with the highest powers for mathematical investigations. Accordingly, under the assumed name of Dettonville, he published a challenge to the mathematicians of Europe, offering two prizes for the solution of his problems. There were but two candidates for these prizes (Lalouère and Wallis), but several distinguished geometers (Sluse, Ricci, Huyghens, Wren, Fermat), without competing for the prize, took the opportunity of communicating their discoveries, the most remarkable being Wren's discovery of the rectification of the curve. Lalouère's solutions were found erroneous, and Pascal was spared the mortification of seeing his prize adjudged to a Jesuit. Wallis not being more successful, Pascal shortly afterwards published his "History of the Cycloid" in French and Latin, together with his own solutions, also extending Wren's theorem, by reducing the rectification of the curtate and prolate cycloids to elliptic arcs. Lalouère and Wallis both replied with treatises on the curve. Want of space compels me to omit other remarkable properties of the curve, and I shall bring this long note to a close by mentioning Huyghens' discovery of the isochronism of oscillation in a cycloid, whence, in seeking the method of constraining a body to oscillate in a cycloid, he was led to the general theory of evolutes, and to the determination of the evolute of the cycloid.

point connected with a circle rolling on the circumference of another. When the point is on the circumference of the rolling circle, the curve generated is called an *epicycloid* or *hypocycloid*, according as the circle rolls on the exterior or interior of the fixed circle; if the generating point be not on the circumference, the curve is called an *epitrochoid*, or *hypotrochoid*.

Let us take for the axis of x that position of the common diameter of the two circles which passes through the generating point; let CO be any other position of it, Q the generating point; let $CN = a$, $ON = b$, $NCB = \phi$, $PON = \psi$, $OQ = d$; then since $BN = NP$, we have $a\phi = b\psi$; $OQM = 180 - (\phi + \psi)$; and the co-ordinates of Q are



$$y = (a + b)\sin\phi - d\sin(\phi + \psi),$$
$$x = (a + b)\cos\phi - d\cos(\phi + \psi);$$

or if $a + b = mb$,

$$y = mb\sin\phi - d\sin m\phi,$$
$$x = mb\cos\phi - d\cos m\phi.$$

Eliminating ϕ from these equations we obtain the equation of the curve, which is not necessarily transcendental. In fact, when the circumferences of the circles are commensurable, after a certain number of revolutions, the generating point returns to a former position, the curve is closed, and of finite algebraic dimensions; but if they be not commensurable, the generating point will not in any finite number of revolutions return to the same position, and the curve will be transcendental.

To obtain the equations of the epicycloid we have only to make $d = \pm b$, and we have

$$y = b\,(m\sin\phi \pm \sin m\phi),$$
$$x = b\,(m\cos\phi \pm \cos m\phi);$$

the lower sign answers to the case when the axis of x passes through the generating point when it is *on* the fixed circle; the upper sign, when it is at its greatest distance from it.

224. The co-ordinates for the case of the hypotrochoid and hypocycloid are found, as the reader can easily verify, by chang-

ing the sign of b in the equations given above. These will be included in the equations which we shall use, by giving negative values to m, or by supposing $m = -n$, where $n = \frac{a-b}{b}$.

The equations given above, if we alter b into mb, and m into $\frac{1}{m}$, become

$$y = mb\left(\frac{1}{m}\sin\phi + \sin\frac{1}{m}\phi\right),$$

$$x = mb\left(\frac{1}{m}\cos\phi + \cos\frac{1}{m}\phi\right);$$

and making $\phi = m\psi$, we see that these equations belong to the same locus as the preceding. We can thus prove that the same hypocycloid is generated whether we take $b = \frac{a \pm c}{2}$. (Euler de duplici genesi Epicycloidum, Acta. Petrop. 1784, referred to by Peacock, Examples, p. 194.) The hypocycloid, when the radius of the moving circle is greater than that of the fixed circle, may also be generated as an epicycloid, for then $m\left(= -\frac{a-b}{b}\right)$ is positive.

225. Tangents can easily be drawn to these curves, for by the same reasoning as that used in Art. 222 the line NQ is normal to the curve. We can thus see also that when a curve is generated by a point on the circumference of one figure rolling on another, there must be a cusp at every point where the generating point meets the fixed curve. For by this construction at such a point the generating point approaches the fixed curve in the direction of its normal, and recedes from it in the same direction; hence it is a stationary point (Art. 31). An epicycloid then consists of a number of similar portions, each united to the next by a cusp; and the extreme radii, from the centre of the fixed circle to any such portion, are inclined at an angle $= \frac{2b\pi}{a}$. The equations of the tangents to the epi- or hypocycloids admit of being written in a very simple form. For

$$\frac{dy}{dx} = \frac{\cos\phi \pm \cos m\phi}{-(\sin\phi \pm \sin m\phi)} = -\frac{\cos\frac{1}{2}(m+1)\phi}{\sin\frac{1}{2}(m+1)\phi}, \text{ or else } = \frac{\sin\frac{1}{2}(m+1)\phi}{\cos\frac{1}{2}(m+1)\phi}.$$

And attending to the condition that the tangent must pass through the point whose co-ordinates have been given in Art. 223, the equation of the tangent becomes

$$x \cos \tfrac{1}{2}(m+1)\phi + y \sin \tfrac{1}{2}(m+1)\phi = (m+1)\, b \cos \tfrac{1}{2}(m-1)\phi,$$

when the axis passes through the generating point at its greatest distance from the centre of the fixed circle; and

$$x \sin \tfrac{1}{2}(m+1)\phi - y \cos \tfrac{1}{2}(m+1)\phi = (m+1)\, b \sin \tfrac{1}{2}(m-1)\phi,$$

when the axis of x passes through the generating point at its least distance from the centre of the fixed circle.

The equation of the normal in the latter case is in the same manner seen to be

$$x \cos \tfrac{1}{2}(m+1)\phi + y \sin \tfrac{1}{2}(m+1)\phi = (m-1)\, b \cos \tfrac{1}{2}(m-1)\phi.$$

Comparing this with the first form of the equation of the tangent, it follows that *the evolute of an epicycloid is a similar epicycloid*, the radii of the circles being altered in the ratio $\dfrac{m-1}{m+1}$, and the generating point of the evolute being at its greatest distance from the centre of the fixed circle when on the same diameter on which the generating point of the original curve is at its least distance.

The same remarks, of course, apply to the hypocycloid.

226. We give the equations of some of the simplest of these curves. First, if the two circles be equal, $m = 2$, and the problem of determining the epicycloid becomes, to find the envelope of an equation of the form

$$x \cos 3\phi + y \sin 3\phi = 3b \cos \phi\,;$$

but this is equivalent to

$$y \tan^3 \phi + 3(b + x) \tan^2 \phi - 3y \tan \phi + 3b - x = 0,$$

whose envelope is

$$4b^2y^2 = \{y^2 + (x+b)^2\}\, \{y^2 + (x+b)^2 - 4b\,(x+b)\},$$

a curve having a cusp at the point $(y, x + b)$, to which y is a tangent.

Writing the equation in the form

$$(x^2 + y^2 - 3b^2)^2 = 4b^3\,(2x + 3b),$$

we see that the curve is a Cartesian oval, of which the origin is a

triple focus (p. 123), and since it has a cusp it is a cardioide (Art. 218). Hence the evolute of a cardioide is a cardioide.

If the rolling circle have a radius half the other, the equation of the tangent to the epicycloid is of the form

$$x \cos 2\theta + y \sin 2\theta = 4b \cos \theta,$$

belonging to the class whose envelope is given, p. 116; the epicycloid therefore is

$$(x^2 + y^2 - 4b^2)^3 = 108b^4x^2.$$

The equation of the tangent to an epitrochoid is in like manner

$$(b \cos\phi - d \cos m\phi)\, x + (b \sin\phi - d \sin m\phi)\, y$$
$$= \{mb^2 + d^2 - (m+1)\, bd \cos (m-1)\, \phi)\}.$$

Let the circles be equal, and therefore $m = 2$; and this equation becomes one of the class whose envelope is given, p. 116. But it is easier in this case to eliminate between the equations

$$x = 2b \cos\phi + d \cos 2\phi, \quad y = 2b \sin\phi + d \sin 2\phi,$$

whence $$x^2 + y^2 = 4b^2 + d^2 + 4bd \cos\phi;$$

and solving for $\cos\phi$, and substituting in the first equation, we have $$(x^2 + y^2 - 2b^2 - d^2)^2 = 4b^2 (2dx + 2d^2 + b^2),$$

the equation of a Cartesian oval having, as may readily be seen, $(y, x + d)$ for a double point. Hence every *limaçon* may be generated as an epitrochoid.

We add some examples of hypotrochoids and hypocycloids. When the radius of the fixed circle is double that of the rolling circle, $m = -1$, and the co-ordinates of any point on the hypotrochoid are

$$x = b \cos\phi + d \cos\phi, \quad y = b \sin\phi - d \sin\phi;$$

the hypotrochoid is therefore the ellipse

$$\frac{x^2}{(b+d)^2} + \frac{y^2}{(b-d)^2} = 1.$$

When $b = d$, the hypocycloid is the diameter $(y = 0)$ of the fixed circle. When $a = 3b$, we have $m = -2$, and the equation of the tangent to the hypocycloid is of the form

$$x \cos\phi - y \sin\phi = b \cos 3\phi,$$

whose envelope, solved by the same method as in the first example, is

$$(x^2 + y^2)^2 + 8bx^3 - 24bxy^2 + 18b^2 (x^2 + y^2) = 27b^4,$$

a curve of the fourth degree having three cusps, the tangents at which meet at the centre. When $a = 4b$, we have $m = -3$, and the tangent has an equation of the form

$$x \sin\phi + y \cos\phi = 2b \sin 2\phi,$$

an equation already discussed, p. 97, and whose envelope is the well-known curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

227. The equation of the reciprocal of an epicycloid is readily obtained, for the tangent being

$$x \cos\tfrac{1}{2}(m+1)\phi + y \sin\tfrac{1}{2}(m+1)\phi = (m+1) b \cos\tfrac{1}{2}(m-1)\phi,$$

it is plain that the perpendicular on the tangent makes an angle $\frac{1}{2}(m+1)\phi$ with the axis of x, and that its length is $(m+1) b \cos\frac{1}{2}(m-1)\phi$; the locus, therefore, of the foot of this perpendicular is

$$p = (m+1) b \cos\left(\frac{m-1}{m+1}\cdot\omega\right),$$

and the reciprocal curve is

$$\rho \cos\left(\frac{m-1}{m+1}\cdot\omega\right) = (m+1) b.$$

In the original curve

$$\rho^2 = x^2 + y^2 = b^2\{m^2 + 1 + 2m\cos(m-1)\phi\},$$

or

$$\rho^2 = b^2(m-1)^2 + 4mb^2\cos^2\tfrac{1}{2}(m-1)\phi,$$

or

$$\rho^2 = a^2 + \frac{4m}{(m+1)^2}p^2.$$

By the formula $R = \dfrac{\rho d\rho}{dp}$ we have the radius of curvature $= \dfrac{4m}{(m+1)^2}p$.*

228. Another general expression for the radius of curvature in *roulettes* (or curves generated by a point on a rolling curve) may be found as follows: Let P, P′ be two consecutive points of

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* The invention of epicycloids is attributed to the Danish astronomer, Roemer, who, in the year 1674, was led to consider these curves in examining the best form for the teeth of wheels. The rectification of these curves was given by Newton, Principia, Book I. Prop. 49.

the curve, M the point of contact of the rolling with the fixed curve, and R the centre of curvature; then PP′, the element of the arc of the roulette, is = MP.PMP′; but by considering the curves as polygons of an infinite number of sides, we can see that PMP′, the angle through which PM turns, is equal to the sum (or difference) of the angles between two consecutive tangents to the fixed and to the rolling curve. Hence, if $d\sigma$ be the element of the arc of the roulette, ds the common element of the arcs of the fixed and generating curves, ρ and ρ' the radius of curvature of each, we have

$$d\sigma = \mathrm{MP}\left(\frac{ds}{\rho} + \frac{ds}{\rho'}\right);$$

but this element, $d\sigma$, is also equal to PR, the radius of curvature, multiplied by the angle between two consecutive normals; and if we call ϕ the angle OMP, between the normals to the roulette and to the fixed curve, then the angle between two consecutive normals to the roulette is $\dfrac{\cos\phi\, ds}{\mathrm{MR}}$.

Hence

$$\frac{\mathrm{MP}+\mathrm{MR}}{\mathrm{MP}.\mathrm{MR}} = \frac{1}{\cos\phi}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

and

$$\mathrm{PR} = \frac{\mathrm{MP}^2\left(\dfrac{1}{\rho} + \dfrac{1}{\rho'}\right)}{\mathrm{MP}\left(\dfrac{1}{\rho} + \dfrac{1}{\rho'}\right) - \cos\phi}.$$

(See Liouville, vol. x. p. 150.)

229. A large class of transcendental curves is obtained by taking the ordinate some trigonometrical function of the abscissa. There is no difficulty in deriving the shape of such curves from their equation. For example, $y = \sin x$ has positive and constantly increasing ordinates until $x = \frac{\pi}{2}$; the ordinates then decrease in like manner, until $x = \pi$, when the curve crosses the axis at an angle of 45°, and has a similar portion on the negative side of the axis between $x = \pi$ and $x = 2\pi$. The curve, therefore, consists of an infinity of similar portions on alternate sides of the axis.

So again, $y = \tan x$ represents a curve, of which the ordinates

increase regularly from $x = 0$ to $x = \frac{\pi}{2}$, when y is infinite, and the line $x = \frac{\pi}{2}$ an asymptote. For greater values of x, y alters from negative infinity to 0, when $x = \pi$. The curve then consists of an infinity of infinite branches, having an infinity of asymptotes, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, &c., and, as may be readily seen, points of inflexion at $x = 0$, $x = \pi$, $x = 2\pi$, &c.

In like manner the reader may discuss the figure of $y = \sec x$, which also consists of a number of infinite branches, only that each branch, instead of crossing the axis, as in the last case, lies altogether at the same side of it. The branches lie alternately on the positive and negative sides of the axis of x. To the same family belongs a curve called the *companion to the cycloid.* It is generated by producing the ordinates of a circle, not as in the case of the cycloid, until the *produced part* be equal to the arc, but until the entire be equal to the arc. If then the centre be the origin, the curve is represented by the equations

$$x = a\cos\theta,\quad y = a\theta,\quad x = a\cos\frac{y}{a};$$

a curve of the same family as the curve of sines.

230. Next after curves depending on trigonometrical, we may mention those depending on exponential functions. The *logarithmic curve* is characterized by the property that the abscissa is proportional to the logarithm of the ordinate; and its equation therefore is

$$x = m\log y,\quad \text{or}\quad y = a^x.$$

The curve then has the axis of x for an asymptote, since, if $x = -\infty$, $y = 0$; it cuts the axis of y at a distance equal to the unit of length, and then increases to positive infinity. The subtangent of the logarithmic curve is constant; for its value, being in general $\frac{ydx}{dy}$, becomes for this curve $= m$.

Some controversy has arisen as to the proper interpretation of the equation of this curve, $y = e^x$. Attention was at first only paid to the branch of the curve on the positive side of the axis of x,

arising from taking the single real positive value of e^x which corresponds to every value of x. Euler, in his Analysis Infinitorum, II. p. 290, pointed out the necessity of attending to the multiplicity of values which the function admits of; and the same subject has been more fully developed by M. Vincent. (Gergonne's Annales, vol. xv. p. 1.) Thus, if x be any fraction with an even denominator, e^x has a real negative as well as a positive value, and therefore there must be a point corresponding to this value of x on the negative side of the axis, but there is no continuous branch on that side of the axis, since, when x is a fraction with an odd denominator, e^x can have only a real positive value. The general expression, including all values of the ordinate, is found by multiplying the numerical expression for e^x, by the imaginary roots of unity, whose general expression is $\cos 2mx\pi + \sqrt{-1} \sin 2mx\pi$, where m must be made to receive in succession every integer value. This is equivalent to saying that the equation $y = e^x$ must be considered as representing not only one real branch, but also an infinity of imaginary branches included in the formula $y = e^{2m\pi x\sqrt{-1}}$. Any one of these imaginary branches contains a number of real points, where it meets the branch $y = e^{-2m\pi x\sqrt{-1}}$, and which must be considered as conjugate points on the curve. There are an infinity of such points, all lying either on the real branch of the curve, or on the similar branch on the negative side of the axis of x. The latter branch is curious, since, though every point of it may be considered as belonging to the logarithmic curve, no two points of it are consecutive to each other, for two consecutive points will belong to different branches. There is thus formed what M. Vincent calls a "courbe pointillée." In one point, however, M. Vincent appears to me to have fallen into a grave error. He says that the points of this branch are to be carefully distinguished from conjugate points; for that at a conjugate point the differential coefficients have imaginary values, but that at one of these points, on the negative side of the axis, the differential coefficients, being all equal to e^x, are all real, and only differ in sign from those of the corresponding points on the positive side of the axis. It is truly astonishing that M. Vincent should have failed to observe, that if the differential coefficients were all real, it would follow from Taylor's theorem that the next consecutive point must be a real point on the curve, and so that

the negative branch would be an ordinary branch of the curve. But in fact, any one of these negative points must be considered as belonging to a branch whose equation is of the form $y = e^{2m\pi x\sqrt{-1}}$, and the corresponding differential coefficient will be $2m\pi\sqrt{-1}\,y$. These points, then, arising from the intersection of two imaginary branches, answer in every respect to the description of conjugate points.*

231. It appears natural to take the *catenary* next in order, whose equation resembles that of the logarithmic in form; although the discussion of it will oblige us to anticipate some topics which we had intended reserving for the last Chapter. This curve is the form assumed by an inelastic chain of uniform density when left at rest. Very simple mechanical considerations lead to the property, which we shall take as the mathematical definition of the curve, viz., that the arc, measured from the lowest point, is proportional to the tangent of the angle made with the horizontal tangent, by the tangent at the upper extremity. If then the axes be a vertical through the lowest point, and a horizontal line, we have $s = h\dfrac{dy}{dx}$. Now to rectangular axes the element of the arc is the base of a right-angled triangle, of which dx and dy are the sides, or $ds^2 = dx^2 + dy^2$. By the equation of the curve we shall have, therefore,

$$s^2 + h^2 = h^2\frac{ds^2}{dx^2}, \qquad dx = \frac{hds}{\sqrt{(s^2 + h^2)}},$$

$$\frac{x}{h} = \log\left\{\frac{s + \sqrt{(s^2 + h^2)}}{h}\right\},$$

* Some objections to M. Vincent's views, which are worth being considered, will be found in a paper by Mr. Gregory, Cambridge Mathematical Jour., vol. i. pp. 231, 264. But I do not think that the punctuated curve, as presented here, offers any insurmountable difficulty. In all our dealings with infinity we must expect anomalies. We have seen, in the first Article of this Chapter, that a transcendental curve, in possessing an infinity of conjugate points, only exercises an undoubted privilege, nor have we any just ground of complaint, though these conjugate points be placed indefinitely near to one another, and be ranged on a certain locus. Thus (Art. 225) every point of the base of a transcendental epicycloid is a cusp on the curve, though no two points of it are consecutive to each other. In like manner a transcendental epitrochoid has an infinity of non-consecutive double points ranged on a circle; and these are conjugate points when the generating point is within the moving circle. This illustration I owe to Dr. Hart.

the constant being taken so that s and x shall vanish together. Hence

$$e^{\frac{x}{h}} + e^{-\frac{x}{h}} = \frac{2\sqrt{(s^2 + h^2)}}{h}; \quad e^{\frac{x}{h}} - e^{-\frac{x}{h}} = \frac{2s}{h}.$$

But in like manner the equation of the curve gives

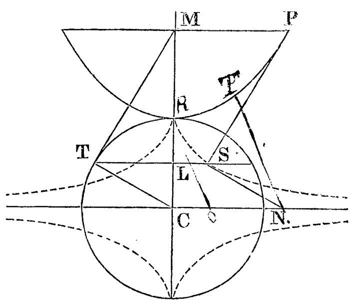
$$\frac{s^2 + h^2}{s^2} = \frac{ds^2}{dy^2}; \quad dy = \frac{sds}{\sqrt{(s^2 + h^2)}}.$$

Hence $y^2 = s^2 + h^2$, provided we suppose the axes so taken that when s or $x = 0$, y shall be $= h$. This value of y gives at once the equation of the curve, viz.:

$$y = \frac{h}{2}\left(e^{\frac{x}{h}} + e^{-\frac{x}{h}}\right).$$

232. We get from this equation

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right) = \frac{s}{h} = \frac{\sqrt{(y^2 - h^2)}}{h}.$$



Hence we are led to the following construction. From the foot of the ordinate M draw the tangent MT to the circle described with the centre C and radius h; then $MC = y$, $CT = h$, $MT = \sqrt{(y^2 - h^2)}$; $\tan MCT = \tan MTL = \frac{\sqrt{(y^2 - h^2)}}{h}$; hence the tangent PS is parallel to MT. The same values prove also that PS = MT = the arc from P to the lowest point. The locus of the point S is therefore the involute of the catenary, and SN is its tangent, since PS must be normal to the locus of S, being tangent to its evolute. The involute of the catenary is therefore a curve such that the intercept SN, on its tangent between the point of contact and a fixed right line, is constant.* Such a curve is called the *tractrix*.

* The form of equilibrium of a flexible chain was first investigated by Galileo, who pronounced the curve to be a parabola. His error was detected experimentally in 1669 by Joachim Jungius, a German geometer: but the true form of the catenary was only ob-

Draw TN normal & RO parallel to TN, & show CO = RT

233. The equation of the tractrix can be obtained without much difficulty. For the length between the foot of the ordinate from S and the point N is $\sqrt{(h^2 - y^2)}$; it also is, by making $y = 0$ in the equation of the tangent, $-\frac{ydx}{dy}$. Hence the differential equation of the curve is

$$-\frac{ydx}{dy} = \sqrt{(h^2 - y^2)},$$

which at once is made rational by putting $z^2 = h^2 - y^2$, and gives

$$dx = \frac{h^2 dz}{h^2 - z^2} - dz.$$

We have then

$$x = h \log \left(\frac{h + \sqrt{(h^2 - y^2)}}{y} \right) - \sqrt{(h^2 - y^2)}.$$

It will be readily seen that the curve consists of four similar portions, as in the dotted curve on the figure; and the construction of the last Article shows at once geometrically how to draw a tangent to the curve.

The *syntractrix* is the locus of a point Q on the tangent to the tractrix, which divides into portions of given length the constant line SN. Let the co-ordinates of the point on the tractrix be $x'y'$, of those on the required locus xy; let the length QN $= k$, then we shall have $ky' = hy$; and

$$\sqrt{(h^2 - y'^2)} - \sqrt{(k^2 - y^2)} = x - x';$$

and since, by the equation of the tractrix,

$$x' + \sqrt{(h^2 - y'^2)} = h \log \left\{ \frac{h + \sqrt{(h^2 - y'^2)}}{y'} \right\},$$

that of the syntractrix will be

$$x + \sqrt{(k^2 - y^2)} = h \log \left\{ \frac{k + \sqrt{(k^2 - y^2)}}{y} \right\}.$$

234. The tractrix is a particular case of the general problem of equi-tangential curves, where it is required to find a curve such that the intercept on the tangent between the curve and a fixed

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tained by James Bernouilli in 1691. Gregory (in his Examples, p. 134) refers to what would seem to be an interesting memoir by Professor Wallace on this curve (Edinburgh Transactions, vol. xiv. p. 625).

directrix shall be constant. We may next mention the problem of "curves of pursuit," which we cannot better lay before the reader than by copying the passage where the question seems to have been first proposed.

"M. Dubois-Aymé se promenait sur le bord de la mer; il aperçut à quelque distance, quelqu'un de sa connaissance, et se mit à courir pour l'atteindre; son chien, qui s'était écarté, courut vers lui en décrivant une courbe dont l'empreinte resta sur le sable. M. Dubois, revenant sur ses pas, fut frappé de la régularité de cette courbe, et il en chercha l'équation, en supposant, 1°, que le chien se dirigeait constamment vers l'éndroit où il voyait son maître; 2°, que le maître parcourait une ligne droite; 3°, que les vitesses du maître et du chien étaient uniformes."*

The correct solution of the problem appears to have been first given by M. de St. Laurent, Gergonne's Annales, vol. xiii. p. 145. The intercept made by the tangent on the axis of y is $y - x\dfrac{dy}{dx}$, and by hypothesis the increment of this is to be proportional to the increment of the arc, or putting $\dfrac{dy}{dx} = p$,

$$-xdp = h\sqrt{(1+p^2)}\,dx,$$

$$\log x^h + \log\{p + \sqrt{(1+p^2)} + \log \mathrm{A} = 0,$$

$$2p = \mathrm{A}^{-1}x^{-h} - \mathrm{A}x^h,$$

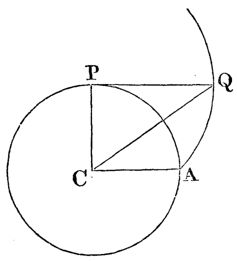
$$2y = \mathrm{C} - \frac{\mathrm{A}}{h+1}x^{h+1} - \frac{\mathrm{A}^{-1}}{h-1}x^{-h+1}.$$

This curve will then be algebraic, except in the case when $h = 1$, when we have to substitute $\log x$ for $\dfrac{x^{-h+1}}{h-1}$.

235. The *involute of the circle* is another transcendental curve whose equation can be obtained without much difficulty. This is equivalent to the following problem: "If on the tangent at any point P of a circle there be taken a portion, PQ, such that it shall be equal to the arc AP measured from any fixed point A; to find the locus of Q." Let the radius of the circle $= a$, the centre

* Correspondance sur l'école polytechnique, ii. 275.

being C, and the radius vector CQ = ρ; let PCA = ϕ, QCA = θ. Then PQ = $\sqrt{(\rho^2 - a^2)}$; and it also = $a\phi$ by hypothesis; but



$$\phi = \theta + \cos^{-1}\frac{a}{\rho}.$$

Hence the polar equation of the locus is

$$\frac{\sqrt{(\rho^2 - a^2)}}{a} = \theta + \cos^{-1}\frac{a}{\rho}.$$

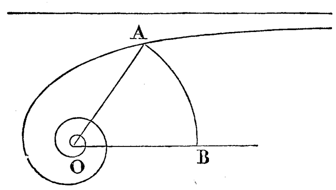
The involute of the circle is the locus of the intersection of tangents drawn at the points where any ordinate meets a circle and the corresponding cycloid.

236. We shall conclude this Chapter with some account of spirals. In these curves referred to polar co-ordinates, the radius vector is not a periodic function of the angle, but one which gives an infinity of different values when we substitute $\omega = \theta$, $\omega = 2\pi + \theta$, $\omega = 4\pi + \theta$, &c. The same right line then meets the curve in an infinity of points, and the curve is transcendental. Let us first take the *spiral of Archimedes*, which is the path described by a point receding uniformly from the origin, while the radius vector on which it travels moves also uniformly round the origin. The polar equation of the curve is then

$$\rho = a\omega.$$

This spiral is the locus of the foot of the perpendicular on the tangent to the involute discussed in the last Article. For, from the nature of evolutes, the tangent to the locus of Q is perpendicular to PQ; and the length of the perpendicular on that tangent from C will = PQ = $a\phi$, and ϕ is proportional to the angle this perpendicular makes with a fixed line. Hence, too, the reciprocal of the involute is the *hyperbolic spiral* $\rho\omega = a$, which we shall discuss in the next Article. The spiral of Archimedes is one of a family included in the general equation $\rho = a\omega^n$, in all which the tangent approaches more nearly to being perpendicular to the radius vector the further the point recedes from the origin. For $\frac{\rho d\omega}{d\rho} = \frac{\omega}{n}$; therefore (p. 102) the tangent of the angle made by the radius vector with the tangent increases as ω increases, but does not actually become infinite until ω is infinite.

237. We have just mentioned the equation of the hyperbolic spiral $\rho\omega = a$. This spiral has an asymptote parallel to the line from which ω is measured; for the perpendicular from any point of the spiral on this line is $\rho \sin \omega = \frac{a \sin \omega}{\omega}$, which, when ω vanishes, and ρ becomes infinite, has the finite value a. Or again, we might calculate the length of the perpendicular from the origin on the tangent. The tangent of the angle made by the radius vector with the tangent is $\frac{\rho d\omega}{d\rho} = -\omega$; hence the perpendicular is $\frac{a\rho}{\sqrt{(a^2+\rho^2)}}$, which, when ρ becomes infinite, is $= a$. The form of the curve is then as here given. The polar subtangent of the hyperbolic spiral is constant. The arc AB of the circle described with the radius OA to any point of the curve is obviously constant.



Another spiral worth mentioning is the *lituus* $\rho^2\omega = a^2$; this also has an asymptote, viz., the line from which ω is measured; for the distance of any point of it from this line, $\rho \sin \omega = \frac{a^2 \sin \omega}{\rho\omega}$, decreases indefinitely as ρ increases, and ω consequently diminishes.

238. We shall mention in the last place the *logarithmic spiral*, $\rho = a^\omega$. In this curve ρ increases indefinitely with ω; when ω is 0 it $= 1$, and diminishes further for negative values of ω, but it does not vanish until ω becomes negative infinity; hence the curve has an infinity of convolutions before reaching the pole. One of the fundamental properties of this curve is, that it cuts all the radii vectores at a constant angle, for $\frac{\rho d\omega}{d\rho}$ becomes the modulus of the system of logarithms which has a for its base; the angle, therefore, made by the radius vector with the tangent always has this modulus for its tangent. From this property we at once obtain the rectification of the curve; for if we consider

the elementary triangle which has the element of the arc for its hypothenuse, and the increment of the radius vector for one side, we see that the element of the arc is equal to the increment of the radius vector multiplied by the secant of this constant angle, and hence that any arc is equal to the difference of the extreme radii vectores multiplied by the secant of the same angle. The entire length, measured from any point P to the pole being $\rho \sec\theta$, is constructed by erecting at the pole OQ perpendicular to OP; PQ will then be the required length. The locus of Q will evidently be an involute of the curve, but the angles of the triangle OPQ being constant, OQ is proportional to OP; and it makes with OP a right angle, the locus of Q is therefore also a logarithmic spiral, constructed by turning round the radii vectores of the given curve through a right angle, and altering them in a fixed ratio. Conversely the evolute of a logarithmic spiral is a logarithmic spiral. The locus of the foot of the perpendicular on the tangent is likewise a logarithmic spiral, for it also bears a fixed ratio to the radius vector, and makes with it a constant angle. The caustics by reflexion and refraction, the light being incident from the pole, are likewise logarithmic spirals.*

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* The logarithmic spiral was imagined by Des Cartes, and some of its properties discovered by him. The properties of its reproducing itself in various ways, as stated above, were discovered by James Bernouilli, and excited his warm admiration. The conclusion of his paper on this curve (in the Acta Eruditorum, An. 1692, p. 212) has been cited by Dean Peacock in his Examples, and copied by more than one other writer, but I cannot bring myself to omit so fine a specimen of mathematic enthusiasm.

"Cum autem ob proprietatem tam singularem tamque admirabilem mire mihi placeat spira hæc mirabilis, sic ut ejus contemplatione satiari vix queam; cogitavi, illam ad varias res symbolice repræsentandas non inconcinne adhiberi posse. Quoniam enim semper sibi similem et eandem spiram gignit, utcunque volvatur, evolvatur, radiet; hinc poterit esse vel sobolis parentibus per omnia similis emblema; *simillima filia matri*: vel (si rem æternæ veritatis Fidei mysteriis accommodare non est prohibitum) ipsius æternæ generationis Filii, qui Patris veluti imago, et ab illo ut lumen a lumine emanans, eidem ὁμοούσιος existit, qualiscunque adumbratio. Aut, si mavis, quia curva nostra mirabilis in ipsa mutatione semper sibi constantissime manet similis et numero eadem, poterit esse vel fortitudinis et constantiæ in adversitatibus; vel etiam carnis nostræ post varias alterationes et tandem ipsam quoque mortem, ejusdem numero resurrecturæ symbolum; adeo quidem ut si Archimedem imitandi hodienum consuetudo obtineret, libenter spiram hanc tumulo meo juberem incidi cum Epigraphe: *Eadem numero mutata resurget.*"

CHAPTER VI.

GENERAL METHODS.

TRANSFORMATION OF CURVES.

239. We purpose to devote this Chapter to an account of some general methods, by which the properties of one curve may be derived from those of another. Of this kind are the methods of projections and of reciprocal polars, explained in the former part of this treatise, but we now mean to resume the subject from a general analytical point of view. If in the equations of any number of curves we substitute ξ, η, ζ, for x, y, z respectively, $\xi\eta\zeta$ being any functions of the co-ordinates, we shall obtain a new system of curves corresponding each to one of the first system. We shall first confine our attention to the case of *collinear transformation*, when $\xi\eta\zeta$ represent right lines, when, consequently, $\phi(\xi\eta\zeta)$ always represents a curve of the same degree as $\phi(xyz)$; when, therefore, every right line of the first system has a right line corresponding in the second, and every point of the first system has one, and but one, point corresponding to it in the second. Since $\xi\eta\zeta$, expressed in terms of xyz, contain each three constants, there are nine constants employed in the entire transformation; but since the new curve will be the same if its equation be divided by any constant, the method of collinear transformation can only be considered as involving eight arbitrary constants.

240. *To a pencil of four right lines meeting in a point corresponds a pencil whose anharmonic ratio is the same.*

For we have seen (*Conics*, p. 52) that the anharmonic ratio of the pencil $L - aM$, $L - bM$, $L - cM$, $L - dM$, is $\dfrac{(a-b)(c-d)}{(a-c)(b-d)}$, a quantity not depending on the functions L, M; but by the

transformation in question these four lines become $\lambda - a\mu$, $\lambda - b\mu$, $\lambda - c\mu$, $\lambda - d\mu$, where λ, μ are what L, M become when $\xi\eta\zeta$ are substituted for xyz; these represent four lines meeting in a point whose anharmonic function is the same as that of the first pencil. It follows then, also, that to four points on a right line correspond four other points on a right line whose anharmonic function is the same.

241. These results hold whether or not the lines xyz, $\xi\eta\zeta$ be in the same plane. If they be in the same plane, it is to be remarked that to every line will in general correspond a different one, according as the first line is considered as belonging to the first or the second system. Thus to the line ξ of the second system corresponds the line x of the first; but if ξ were considered as belonging to the first system, its equation must admit of being expressed in the form $ax + by + cz = 0$, and then to it would correspond, in the second system, the line $a\xi + b\eta + c\zeta$, a line which will in general be different from the line x.

We can give a geometrical construction for the process of finding the line corresponding to a given one. First, let the line pass through one of the points xy. Then to any line $Ax + By$ corresponds $A\xi + B\eta$, which must intersect the former on the conic $x\eta = \xi y$. This conic is given; hence to find the line corresponding to one which passes through the point xy we have only to join the point where the given line meets this given conic to the point $\xi\eta$. Similarly the line corresponding to one through xz, or through yz, is found by joining the point where this line cuts $x\xi = \xi z$, or $y\zeta = z\eta$, to $\xi\zeta$ or $\eta\zeta$.

To find now the point corresponding to any given one, we have only to join the given point to xy and to xz, and to construct the two corresponding lines, whose intersection will give the corresponding point required. To construct the line corresponding to any given one, we have only to construct the points corresponding to any two points of it (which may, for simplicity, be taken on x, y, or z), and the line joining those will be the required line.

242. It appears from this construction that *there are three*

points which are the same for both systems ; for these are three points common to the three conics (p. 157),

$$\frac{x}{\xi}=\frac{y}{\eta}=\frac{z}{\zeta};$$

and from the foregoing construction it appears that to the line joining one of these points to xy answers the line joining it to $\xi\eta$, and to the line joining it to xz answers the line joining it to $\xi\zeta$; the point of intersection, therefore, is the same for both systems.

The lines joining the three points are of course also the same for both systems.

243. Let a, b, c be any three lines in one system, α, β, γ the corresponding lines in the other system; then to any line, $Aa + Bb + Cc$, of the one system will correspond $A\alpha + B\beta + C\gamma$ of the other. For let

$$\begin{aligned}
a &= Lx + My + Nz, & \alpha &= L\xi + M\eta + N\zeta,\\
b &= L_2x + M_2y + N_2z, & \beta &= L_2\xi + M_2\eta + N_2\zeta,\\
c &= L_3x + M_3y + N_3z, & \gamma &= L_3\xi + M_3\eta + N_3\zeta,
\end{aligned}$$

then it is evident that the equations which express x, y, z in terms of a, b, c, and ξ, η, ζ in terms of α, β, γ, will have the same coefficients, and, therefore, that if the equation of any curve be transformed from a function of xyz to a function of abc, the corresponding curve, when transformed from a function of $\xi\eta\zeta$ to a function of $\alpha\beta\gamma$, will still have the same coefficients as the other.

Hence, then, if α, β, γ be the three lines whose position is the same for both systems, the transformation of this method must simply be to alter the equation of any curve of the first system, $\phi(\alpha,\beta,\gamma)=0$, into $\phi(l\alpha, m\beta, n\gamma)=0$, when we shall have the corresponding curve of the other system.

244. The method of projections is, as we have already remarked, a case of this collinear transformation. In this method the line joining any two corresponding points passes through a fixed point, viz., the vertex of the projecting cone; and any two corresponding lines intersect on a certain fixed line, viz., the intersection of the two planes of section. If one of the planes were turned about this line so as to be brought to coincide with the other, the figures would still have the property that the line join-

ing two corresponding points would pass through a fixed point; for consider the triangles formed by three pair of corresponding lines; and since the corresponding sides intersect in a right line, the lines joining corresponding vertices meet in a point. It is easy to form the most general equation of such a system. Let $ax + by + cz = 0$ be the equation of the line on which the corresponding lines intersect, then it is evident that the equations of $\xi\eta\zeta$ (the lines corresponding to xyz) will be of the form

$$\xi = a'x + by + cz = 0,$$
$$\eta = ax + b'y + cz = 0,$$
$$\zeta = ax + by + c'z = 0,$$

a system involving three constants less than in the general case, and therefore only five in all.

We shall call the point at which the lines joining corresponding points meet, the *pole* of the system, and the line on which corresponding lines intersect, the *axis* of the system. By subtracting successively each pair of the equations just written, it will be seen that the pole of the system whose equations we have written is given by the equations

$$(a - a')\,x = (b - b')\,y = (c - c')\,z.$$

The simplest forms of the equations of projective transformation are derived as follows: Any line passing through the pole is the same for the new figure; for any two points of it have corresponding to them two points on the same line. Hence if the pole be taken at the point xy, the two lines x and y are unaltered by transformation; and any other line, $Ax + By + Cz = 0$, has corresponding to it, $Ax + By + C\zeta = 0$, the two lines intersecting on the fixed axis, $z - \zeta = 0$. Any line $Ax + By = 0$ passing through the pole evidently remains unchanged.

245. Conversely, if two collinear figures in the same plane have the property that any corresponding lines intersect on a fixed axis, one of the figures may be considered as a projection of the other. For let the plane of one of the figures be turned round this axis, and consider any three pairs of corresponding points, ABC, *abc*, the corresponding sides of these triangles intersecting in L, M, N. Then when the plane is turned round, A*a*, B*b*

must still intersect (since the lines AB, ab intersect in N, and are therefore in the same plane); and by the theory of transversals Bb cuts Aa when produced in the same ratio as before the figures were turned round. But in like manner Cc, and the line joining any other pair of corresponding points, meets Aa in the very same point.

246. The general collinear method of transformation, containing three constants more than the projective method, appears at first sight a more powerful instrument of research, and we should expect to arrive, by its means, at extensions of known theorems more general than those with which the method of projections had furnished us. It is obvious, however, that if a figure were transferred bodily to some other position, we should have a collinear transformation, in which to every line of the first figure would correspond a line of the second figure, but yet which would give us no new geometrical information. Now we owe to M. Magnus the remark, that the most general collinear transformation may be reduced to a projective transformation by turning the figure round a given angle, and then moving it for a given length along a given direction; these three latter constants being just the number by which the collinear transformation appears to be more general than the projective.

To see this, we must first observe, that if a figure be moved in any direction without twisting, since all lines remain parallel to their first position, the position of every point at infinity remains unaffected by the operation.

Next, let the whole figure be made to turn round any fixed point, and any systems of parallel lines will still remain a system of parallel lines, although no longer parallel to its former direction; hence any point at infinity will still remain at infinity, and therefore the line at infinity is the same for the figure in both its functions. Moreover, since any circle will remain a circle, however it be moved, we see that the two circular points at infinity will not be disturbed, no matter how the figure be moved.

If then it be required to move a figure so as to have a projective position with a given collinear figure, let the two circular points be ω, ω', the two corresponding points of the second figure

o, o', since no motion of the first figure can alter the position of ω and ω', the only possible position of the required pole of the two figures is the point λ, where the lines $o\omega$, $o'\omega'$ intersect. Let then the first figure be moved so as to bring the point l, which corresponds to λ, to coincide with it. Moreover, let the first figure be turned about l so as to bring m, μ (any other pair of corresponding points) into a line with l; then we say that the two figures will have a projective position, and the line joining any other two corresponding points, n, ν, must also pass through l. For the anharmonic ratio of $\{l.\omega\omega'\mu\nu\} = \{l.oo'mn\}$ (Art. 240), and since three lines of the system are the same for both, the fourth must also be the same for both. M. Magnus's theorem has then been proved.

247. There is no difficulty in expressing analytically the geometrical theory of the last Article. Thus if it be required to find the co-ordinates of the point l in the case of the general transformation, we are, first, by the theory just laid down, to find the line $o\omega$ joining the point $(x + y\sqrt{-1}, z)$ to

$$[\{ax + by + cz + (a_1x + b_1y + c_1z)\sqrt{-1}\},\ a_2x + b_2y + c_2z],$$

this will be

$$(b_2 - a_2\sqrt{-1})\ \{(ax + by + cz) + (a_1x + b_1y + c_1z)\sqrt{-1}\}$$
$$- \{a_1 + b + (b_1 - a)\sqrt{-1}\}\ (a_2x + b_2y + c_2z) = 0,$$

or $(ab_2 - a_2b)\,x + (a_2b_1 - a_1b_2)\,y + \{(cb_2 - c_2b) + (c_1a_2 - c_2a_1)\}\,z$

$$+\sqrt{-1}\,\{(a_1b_2 - b_1a_2)\,x + (ab_2 - a_2b)\,y + (c_1b_2 - b_1c_2)\,z + (ac_2 - ca_2)\,z\} = 0.$$

The line joining $\omega'o'$ will only differ from this in the sign of the quantity multiplying $\sqrt{-1}$. The point required is therefore the intersection of the two lines found by putting the real and imaginary part of the equation separately $= 0$.

It is not necessary to dwell on particular species of collinear transformation, such, for example, as similarity. We may only mention one kind of collinear relation, in which the area of any space on the one figure is equal to that of the corresponding space on the other figure. It is easy to see that such a transformation is possible. For let the triangle formed by xyz be equal to that formed by $\xi\eta\zeta$, then, if we take any point O on the first figure, it

will be easy to determine a corresponding point o on the second, such that $Oxy = o\xi\eta$ and $Oxz = o\xi\zeta$; and therefore that $Oyz = o\eta\xi$; and the triangle formed by any three points OPQ will be equal to that formed by opq, the corresponding points so determined.

This species of collinear relation differs from orthogonal projection just as the general collinear relation differs from projection in general.

248. Next, let the equation $\phi(xyz) = 0$ be transformed into $\phi(\xi\eta\zeta) = 0$, where xyz are trilinear co-ordinates, and $\xi\eta\zeta$ tangential co-ordinates; then to every point of one system answers a line of the other; and to the line joining two points of one system the intersection of the corresponding lines of the other.

If we wish to employ trilinear co-ordinates only, and to express the general equation of the line corresponding to any given point $x'y'z'$; since the perpendiculars on this line from the points xy, yz, zx are to be proportional to the perpendiculars from the corresponding point on three fixed lines, the equation of that line must be of the form

$$(a_1x' + b_1y' + c_1z')\,x + (a_2x' + b_2y' + c_2z')\,y + (a_3x' + b_3y' + c_3z')\,z = 0.$$

This is an equation involving eight constants, and would coincide with the equation of the polar of a point with regard to a conic section, only if $b_1 = a_2$, $c_1 = a_3$, $b_3 = c_2$; the latter equation involving but five constants.

The anharmonic ratio of any pencil on the one system is equal to that of the corresponding four points of the other system; for $L - aM$, $L - bM$, $L - cM$, $L - dM$, have $\dfrac{(a-b)\,(c-d)}{(a-c)\,(b-d)}$ for their anharmonic ratio, whether L, M mean equations in line or point co-ordinates.

249. In the general case every point has a different line corresponding to it when the point is considered as belonging to the first and to the second system. Thus the equation just written expresses the relation between any point $x'y'z'$ of the first system and any point xyz on a corresponding line of the second system. If now the latter point be fixed, and the first variable, we have,

for the equation of the line of the first system corresponding to any point of the second,

$$(a_1x' + a_2y' + a_3z')\,x + (b_1x' + b_2y' + b_3z')\,y + (c_1x' + c_2y' + c_3z')\,z = 0.$$

In the case of reciprocals, with regard to a conic, the same line corresponds to a point, whether that point be considered as belonging to the first or second system; but this is not true in general.

250. In order to give a geometric construction for the line corresponding to any point, we shall first inquire the locus of the points which lie on their polars. This is obviously

$$a_1x^2 + (a_2 + b_1)\,xy + b_2y^2 + (b_3 + c_2)\,yz + (a_3 + c_1)\,xz + c_3z^2 = \mathrm{U} = 0,$$

and is the same conic whether the point be considered as belonging to the first or to the second system. We shall call this *the pole conic.*

Next let us seek the envelope of lines which pass through their poles. Now the line $ux' + vy' + wz'$ (where $x'y'z'$ is a point on the conic just written) touches (see *Conics*, p. 328)

$$\begin{aligned}&(b_3^2+c_2^2+2b_3c_2-4b_2c_3)\,u^2+(4a_1b_3+4a_1c_2-2a_2a_3-2a_2c_1-2b_1a_3-2b_1c_1)\,wv\\+&(a_3^2+c_1^2+2a_3c_1-4a_1c_3)\,v^2+(4b_2a_3+4b_2c_1-2a_2b_3-2a_2c_2-2b_1b_3-2b_1c_2)\,uw\\+&(a_2^2+b_1^2+2a_2b_1-4a_1b_2)\,w^2+(4a_2c_3+4b_1c_3-2c_1c_2-2a_3b_3-2c_1b_3-2c_2a_3)\,uv\\&=0.\end{aligned}$$

The envelope is therefore a conic, which we shall call *the polar conic,* and which is also the same whether the lines in question belong to the first or to the second system.

Hence, then, we have at once the polar of any point on the *pole conic.* For from that point draw two tangents to the *polar conic,* and one of these is the polar when the given point is considered to belong to the first system; and the other, when it is considered to belong to the second system.

Or conversely, to find the pole of any tangent to the polar conic. We have only to take the two points where this line meets the pole conic, and one of these points is its pole in the first, and the other in the second system. For by the definition the pole must be *on* the given line, and must therefore be a point on the pole conic.

Let it be required now to find the polar of any point O. Draw from it two tangents, OT_1, OT_2, to the polar conic. Let OT_1 meet the pole conic in the points A_1A_2, and let OT_2 meet it in the points B_1B_2. Then if A_1 be the point in the first system which corresponds to OT_1, and B_1 that which corresponds to OT_2, plainly A_1B_1 is the line which corresponds to O. Similarly, A_2B_2 is the other polar of O.

Or, to find the pole of a given line meeting the pole conic in the points AB, from these draw tangents AP_1, AP_2, BQ_1, BQ_2 to the polar conic; and if AP_1, BQ_1 be the lines in the first system, which are the polars of A, B, their intersection gives the point in the first system, which is the pole of AB. And, in like manner, the intersection of AP_2, BQ_2 gives the pole of AB in the second system.

The reader will readily see how these constructions reduce to the ordinary polar reciprocals if $a_2 = b_1$, $b_3 = c_2$, $c_1 = a_3$. The pole and polar conic will then coincide, the polar of any point on that conic is the tangent at that point; and the polar of any other point is the same for both systems, and is the line joining the points of contact of tangents from the point to the conic.

251. It follows at once from these principles that in the general case the pole conic and the polar conic have double contact with each other. For take any point of intersection, its two polars coincide with the tangent at that point to the polar conic; the two poles of this line must therefore coincide, and therefore the two points where it meets the pole conic must coincide, therefore the tangent to the polar conic at their intersection must touch the pole conic also. The same thing is proved for their other point of intersection. Mr. Cayley has proved the same thing analytically, by showing that if $U = 0$ be the equation of the pole conic, that of the polar conic (found by putting for u, v, w their values in the equation of the last Article) may be thrown into the form

$$\{x(a_1b_3 - b_1a_3 + a_2c_1 - a_1c_2) + y(b_2c_1 - b_1c_2 + b_3a_2 - b_2a_3) \\ + z(c_3a_2 - c_2a_3 + c_1b_3 - c_3b_1)\}^2 \\ + 4U.\{a_1(c_2b_3 - b_2c_3) + a_2(b_1c_3 - b_3c_1) + a_3(b_2c_1 - b_1c_2)\} = 0,$$

a form which shows at once that it has double contact with U.

252. There are three points in the general case whose polars are the same with regard to both systems. For let the equations of the polars in each system be

$$ux + vy + wz = 0, \text{ and } u_1x + v_1y + w_1z = 0,$$

then the system of equations

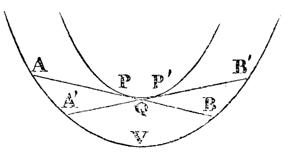
$$\frac{u}{u_1} = \frac{v}{v_1} = \frac{w}{w_1}$$

has already been shown to be satisfied for three points. And the theory laid down in the last Article shows at once what the three points are. For the two points of contact of the pole and polar conics have each the same polar in both systems, viz., the common tangents at these points; and the point at which these tangents intersect has also the same polar in both systems, viz., the chord of contact of the conics.

There are then three points which have the same polar in both systems; and two of these points lie on their polars, but the third does not.

253. It is desirable to show that in the constructions which we have given no ambiguity occurs, and that we need be at no loss to know, of the two poles of a given line, which belongs to the first, and which to the second system.

Since two conics having double contact may always be projected into two similar concentric conics, we use these in the figure for greater simplicity.



Let A, B be the two poles of any tangent to the polar conic, then of the two poles of any other tangent A_1, B_1, A_1 will belong to the first system, since if AB were moved round to coincide with A_1B_1, A would coincide with A_1, and B with B_1. The distinction between the points may be readily made by the help of the following theorem: " A_1B and AB_1 are parallel in the case of two concentric conics; and by the method of projections, in the general case, intersect on the chord of contact of the conics."

Reciprocally, if we draw tangents to the polar conic from two points on the pole conic, we must so number them, oa_1, oa_2, pb_1, pb_2, that the line joining the intersection of oa_1, pb_2 to that of

oa_2, pb_1 may pass through the pole of the chord of contact of the conics.

254. The number of constants in the general case of reciprocals only exceeding by three the number of constants in the case of reciprocals with regard to a conic, it is natural to inquire whether the latter does not only differ from the former by displacement of the figure. It is evident, at any rate, that the general reciprocal $ux + vy + wz$ is only a projection (as in Art. 246) of the reciprocal, with regard to a conic, $x\xi + y\eta + z\zeta$, and that therefore the use of skew reciprocals can lead to no geometric theorem, which we might not obtain by combining the use of ordinary reciprocals with the method of projections.

It is very easy to see what must be the first step, if it be required to move the two figures into such a position that the polar of every point may be the same, no matter to which system that point be considered to belong. For since the position of the line at infinity is unaffected by any displacement of the figure, we must begin by taking its pole in each system, and then moving the systems so that these points shall be brought to coincide. The pole and polar conics will then become concentric and similar, this point being their common centre.

255. Now we say, that if by turning the figures round their common centre O, they can be given such a position that the polar of any point, A, at infinity, shall be the same line, OB, for both systems; then if the polar of any other point, C, at infinity, be the line OD for the first system, it must be also so for the second system. For the anharmonic ratio of the four points of the first system, ABCD, is equal to the corresponding pencil of the second system, viz., OB.OA.OD.OX; and since three legs of this pencil are the same as of that which measures the other, OX must coincide with OC, or the polar of the point D must be the same whether it belong to the first or second system, so also must then the polar of C.

Since now the circular points at infinity are unmoved by any turning of the figure, we have only to take the two polars of either of these points, which in general will not pass through the point, and turn either figure round, so as to bring these polars to

coincide; and then, from what has been just proved, the polars of every other point will coincide.

256. We can readily obtain an expression for the angle through which the figure is to be turned. The two figures being in a concentric position, and the origin being the centre, it is readily seen that the most general equations of the two polars of any point are

$$(a_1x' + b_1y')\,x + (a_2x' + b_2y')\,y + c_3 = 0,$$

and
$$(a_1x' + a_2y')\,x + (b_1x' + b_2y')\,y + c_3 = 0.$$

The two polars of the point at infinity, for which $y' = x'\sqrt{-1}$, are

$$(a_1 + b_1\sqrt{-1})\,x + (a_2 + b_2\sqrt{-1})\,y = 0,$$

and
$$(a_1 + a_2\sqrt{-1})\,x + (b_1 + b_2\sqrt{-1})\,y = 0;$$

and the angle through which one of these lines must be turned to coincide with the other is the difference of the angles whose tangents are

$$-\frac{a_1 + b_1\sqrt{-1}}{a_2 + b_2\sqrt{-1}} \text{ and } -\frac{a_1 + a_2\sqrt{-1}}{b_1 + b_2\sqrt{-1}};$$

but this is the real angle whose tangent is $\dfrac{a_2 - b_1}{a_1 + b_2}$.

257. Or the same result may more simply be obtained as follows: If in general the line of the second system corresponding to the point $x'y'$ in the first, be

$$(a_1x' + b_1y')\,x + (a_2x' + b_2y')\,y + c_3 = 0;$$

then when the second system is turned round an angle θ, the equation of this line will become

$$(a_1x' + b_1y')\,(x\cos\theta - y\sin\theta) + (a_2x' + b_2y')\,(x\sin\theta + y\cos\theta) + c_3 = 0,$$

or
$$\{(a_1\cos\theta + a_2\sin\theta)\,x' + (b_1\cos\theta + b_2\sin\theta)\,y'\}\,x$$
$$+ \{(a_2\cos\theta - a_1\sin\theta)\,x' + (b_2\cos\theta - b_1\sin\theta)\,y'\}\,y + c_3 = 0.$$

But the locus of points of the first system whose polars pass through $x'y'$, that is to say, the line corresponding to $x'y'$, considered as belonging to the transformed system, will be

$$\{(a_1\cos\theta + a_2\sin\theta)\,x' + (a_2\cos\theta - a_1\sin\theta)\,y'\}\,x$$
$$+ \{(b_1\cos\theta + b_2\sin\theta)\,x' + (b_2\cos\theta - b_1\sin\theta)\,y'\}\,y + c_3 = 0.$$

This line will always coincide with the other, if we have

$$b_1 \cos\theta + b_2 \sin\theta = a_2 \cos\theta - a_1 \sin\theta;$$

or, as before,

$$\tan\theta = \frac{a_2 - b_1}{b_2 + a_1}.$$

258. We shall touch but lightly on the more general transformations of $\phi(xyz)$ to $\phi(\xi\eta\zeta)$, where $\xi\eta\zeta$ are any functions of the co-ordinates. It is unnecessary to discuss further the case where $\xi\eta\zeta$ are tangential co-ordinates, since, by the last case, we can first transform, so that $\xi\eta\zeta$ shall be linear functions of these co-ordinates, and afterwards, by the methods now to be explained, transform these to higher functions of the same co-ordinates.

If now we alter $\phi(xyz)$ into $\phi(x^{\frac{1}{2}}, y^{\frac{1}{2}}, z^{\frac{1}{2}})$, or, what is the same thing, $\phi(\xi\eta\zeta)$ into $\phi(\xi^2\eta^2\zeta^2)$, to any point of the one figure will answer four of the other; to any right line, $ax + by + cz$, will answer the conic $ax^{\frac{1}{2}} + by^{\frac{1}{2}} + cz^{\frac{1}{2}}$, touching the sides of the triangle xyz; to any right line $a\xi + b\eta + c\zeta$ will answer the conic $a\xi^2 + b\eta^2 + c\zeta^2$; to any conic $ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ will answer the curve of the fourth degre, $ax + by + cz + 2dy^{\frac{1}{2}}z^{\frac{1}{2}} + 2ez^{\frac{1}{2}}x^{\frac{1}{2}} + 2fx^{\frac{1}{2}}y^{\frac{1}{2}} = 0$. The equation of this conic may be written in the form

$$\frac{x}{d} + \frac{y}{e} + \frac{z}{f} = \left\{\left(\frac{1}{d^2} - \frac{a}{def}\right)x^2 + \left(\frac{1}{e^2} - \frac{b}{def}\right)y^2 + \left(\frac{1}{f^2} - \frac{c}{def}\right)z^2\right\}^{\frac{1}{2}}.$$

It follows, then, that the transformed curve of the fourth degree will be of the form

$$ax^{\frac{1}{2}} + by^{\frac{1}{2}} + cz^{\frac{1}{2}} + dw^{\frac{1}{2}} = 0,$$

having the lines xyz for double tangents, and having three double points. The transformed curve will be a conic if any two of the quantities $d, e, f = 0$.

Thus, then, any theorem concerning points, right lines, and conics may be transformed to a theorem concerning points, conics touching three fixed lines, and curves of the fourth degree having double contact with these lines.

259. Again, if we substitute for $\phi(x, y, z)$, $\phi(x^{-1}, y^{-1}, z^{-1})$, we shall alter any right line, $ax + by + cz$, into a conic $ayz + bzx + cxy$, circumscribing the triangle xyz, and any conic into a curve of the

fourth degree having the vertices of this triangle for double points, viz.,

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz\,(dx + ey + fz) = 0.$$

This curve has three cusps when the conic touches the sides of the triangle xyz.

By these and similar transformations it is easy, from any known theorem, to derive a multitude of new ones; but in the majority of cases the derived theorems are so complicated as to be scarcely worth writing down.

The general transformation of $\phi\,(x^n, y^n, z^n)$ for $\phi\,(xyz)$ assumes a simple form when the line z is at infinity, and x, y pass through the two imaginary circular points. Transforming to polar coordinates, the equations of these two latter lines are

$$\rho\,(\cos\theta \pm \sqrt{-1}\sin\theta) = 0;$$

and when we substitute for these functions their n^{th} powers, it is equivalent to substituting ρ^n for ρ, and $n\theta$ for θ. We are led then to a method of transformation suggested by M. Chasles,* viz., in the polar equation of a curve to substitute ρ^n for ρ, and $n\theta$ for θ.

260. We shall exemplify this method of transformation first in the most simple case. Let us take on every radius vector of a curve a portion equal to its reciprocal, and we transform the equation of any curve $\rho = \phi\theta$ into $\dfrac{1}{\rho} = \phi\theta$. Thus every right line, $\rho\cos\omega = p$, is transformed into a circle through the origin $\rho = p\cos\omega$, and *vice versâ*; any other circle remains a circle (*Conics*, p. 99); the equilateral hyperbola, $\rho^2\cos 2\omega = a^2$, becomes the lemniscata $\rho^2 = a^2\cos 2\omega$; the focal parabola, $\rho^{\frac{1}{2}}\cos\frac{1}{2}\omega = a$, becomes the cardioide $\rho^{\frac{1}{2}} = a\cos\frac{1}{2}\omega$; any conic becomes in gene-

* I ascribe this method of transformation to M. Chasles, because I know that I was familiar with it, and believed it to be his, long before the publication of any of the other papers on the subject cited below. The only writing of M. Chasles I can now refer to is Note xxi. p. 352, of the Aperçu Historique, which, however, only treats of the case where $n = \frac{1}{2}$. I cannot tell whether I was led to the general method as an obvious extension of this, or whether some paper on the subject may not be found among M. Chasles's many contributions to scientific periodicals. If not, this method is to be considered due to Mr. Roberts, who gives several applications of it in a paper of which I have made use here (Liouville, xiii. 209).

ral a curve of the fourth degree, having the two circular infinite points for double points, and also the origin for another; the curve is the *limaçon* if the origin be the focus of the conic; the curve is only of the third degree if the origin be on the conic; it will still have the origin for a double point, and the two circular infinite points will be on the curve.

Ex. 1. "The three points of inflexion of such a curve of the third degree lie on a right line." Hence we deduce: "There are three points on a conic, whose osculating circles pass through a given point on the curve, and these lie on a circle which passes through that point."* The three points will be real for the case of the ellipse, but two imaginary for the hyperbola.

Since a circle is always transformed into a circle by this method, it follows that "if a curve of the fourth degree have the two circular points for double points, and also another double point, then, through any point of the curve can be drawn three osculating circles, whose points of contact lie on a circle through the given point." This theorem, however, may otherwise be obtained, and may be extended, by transforming any curve of the third degree passing through the two infinite circular points.

Ex. 2. "The feet of the perpendiculars on the sides of a triangle from any point on the circumscribing circle lie on one right line." Inversely, "If on three chords of a circle, AB, AC, AD, as diameters, circles be described, the points of intersection of these circles with each other lie on one right line."

Ex. 3. "The circle circumscribing any triangle whose sides touch a parabola, passes through the focus." Inversely, if three circles be described through the cusp to touch a cardioide, their points of intersection with each other lie on one right line.

Ex. 4. "If any right line meet a *limaçon* in four points, the sum of their distances from the pole is constant." Hence, "if a circle through the focus meet a conic in four points, the sum of the reciprocals of their distances from the focus is constant."

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* This theorem was given without demonstration by M. Steiner (Crelle, xxxii. 300); the proof here given was communicated to me by Mr. Ingram. Another by M. Joachinstal will be found in Crelle, xxxvi. 95, where he also proves that the centre of gravity of these three points is the centre of the ellipse.

To the foot of the perpendicular on a tangent will correspond the extremity of the diameter through the origin, of a circle described through it to touch the curve, and the distance of this point from the origin being double that of the centre of the circle, we can deduce from the locus of the foot of the perpendicular on the tangent to one curve, the locus of the centres of circles described through the origin to touch the inverse curve. Thus from the theorem that the locus of the foot of the perpendicular from the focus on the tangent of a conic is a circle, we deduce (as Mr. Stubbs has pointed out) "the locus of the centre of the circle described through the double point of the *limaçon* to touch the curve, is a circle;" or otherwise, "the *limaçon* may be generated as the envelope of circles passing through a given point, and having their centres on a given circle." Generally the reciprocal of a curve is the locus of the centres of circles through the origin to touch the inverse curve. Thus, "the envelope of a circle passing through a given point, and having its centre on a parabola, is a circular cubic with a double point: and having its centre on any conic, is a bicircular curve of the fourth degree having a finite double point."

261. We proceed to add illustrations of the general case of M. Chasles's transformations; viz., when $\rho = \phi\omega$ is altered to $\rho^n = \phi(n\omega)$. It is unnecessary to write down the effect of the transformation for curves of the form $\rho^m = a \cos m\omega$, which are always altered to curves of the same family. A circle becomes an oval of Cassini when $n = 2$, and a *limaçon* when $n = \frac{1}{2}$. Hence, from Example 3 of the last Article, we have: "If three A's be described to touch a B, their points of intersection with each other lie on a C."

1. A, right line; B, focal parabola; C, circle through origin.
2. A, circle; B, cardioide; C, right line.
3. A, central equilateral hyperbola; B, right line; C, lemniscata.
4. A, lemniscata; B, circle through origin; C, central equilateral hyperbola.

Again, Ex. 4 of the last Article gives us: "If an A meet a B in

four points, the sum of the x^{th} powers of their distances from the origin is constant."

1. A, right line; B, cardioide; x, 1.
2. A, circle through origin; B, focal parabola; x, -1.
3. A, central equilateral hyperbola; B, circle through origin; x, 2.
4. A, lemniscata; B, right line; x, -2, &c.

262. Mr. Roberts has noticed that this method of transformation gives us theorems as to the angles at which curves intersect each other. For the tangent of the angle made by the radius vector with the tangent to a curve is (Art. 160) $\dfrac{\rho d\omega}{d\rho}$; but this is unaltered when we substitute $nd\omega$ for $d\omega$, and $\dfrac{nd\rho}{\rho}$ for $\dfrac{d\rho}{\rho}$. Since then the angle between the tangent to one curve and the radius vector is unaltered by transformation, the angle which two curves of one system make with each other will be also unaltered.

The following examples are a few of several given in Mr. Roberts' memoir:

Ex. 1. "A system of parallel lines makes a constant angle with another system of parallel lines." Hence the angle at which two concentric equilateral hyperbolæ cut each other is constant, if the angle between their axes be constant; or generally any curve whose equation is of the form $\rho^n = a^n \cos n\omega$ cuts another whose equation is $\rho^n = b^n \cos n(\omega - \alpha)$ at a constant angle

Ex. 2. "Given base and vertical angle of a triangle, the locus of vertex is a circle." Hence, "If two concentric equilateral hyperbolæ pass each through a fixed point, and cut at a given angle, the locus of intersection is an oval of a Cassini." If the curves be focal parabolæ, the locus of intersection is a *limaçon*.

Tangents at the extremities of a focal chord of a parabola cut at right angles; the same will consequently be true for the cardioide.

A system of concentric circles are cut orthogonally by a system of right lines passing through the common centre. Hence a system of confocal Cassinoids are cut orthogonally by a system of concentric equilateral hyperbolæ through the common foci. In

like manner can be found a system of *limaçons* cut orthogonally by a system of focal parabolæ.*

GEOMETRICAL METHODS.

263. The present seems a convenient place for adding a few useful geometrical theorems.

Among the methods imagined for drawing tangents to curves, before the invention of the differential calculus, that of Roberval, in which may be traced the germ of the method of fluxions, deserves to be mentioned. He considers a curve as generated by the motion of a point, and endeavours to resolve this motion into two others, such that the ratio of these components shall be known; then the diagonal of the parallelogram formed by them will be the tangent to the curve. One or two examples will serve to illustrate the spirit of his method. The point which describes a cycloid has two motions, one progressive, parallel to the base, another rotatory round the generating circle; and the velocities of these motions are equal, since the point will move round the entire circle in the same time that it moves through a space equal to the circumference along the base. Hence, if through a point on the curve we draw a parallel to the base, and a tangent to the generating circle when passing through the point, and if on these lines we take equal portions, and complete the parallelogram of which these are the sides, the diagonal will be the tangent. Similarly for the curtate and prolate cycloids, the only difference being that the length measured along the tangent is now to be *in a given ratio* to that measured along the parallel to base. In like manner the point describing an epicycloid has two motions, one

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* The principal part of the theory of transformations, given above, is taken from Plücker's System der Analytischen Geometrie, Abschnitt i. § 3. I have also made use of Mr. Cayley's papers on the same subject; Liouville, xiv. 40, xv. 351; Cambridge and Dublin Math. Jour., iii. 173; and of M. Chasles's and Mr. Roberts' memoirs, already cited. Mr. Stubbs published a Paper on Inverse Curves in Brewster's Philosophical Magazine, vol. xxiii. The reader will also find a variety of examples of the inverse transformation in papers with which I have lately become acquainted, published by Mr. Stubbs and Mr. Ingram in the Transactions of the Dublin Philosophical Society. The method which is the subject of Art 264 is taken from Chasles, Aperçu, p. 548. See also Liouville, vol. x. 148, 264, where the same method is applied to finding centres of curvature. Art. 267 is written from my recollection of some articles in Poncelet's Projective Properties.

round the circumference of the moving circle, the other round the centre of the fixed circle; therefore we are to take equal portions on a tangent to the moving circle, and on a perpendicular to the line joining the centres, and complete the parallelogram as before. So again, for a point describing an hyperbola or an ellipse, the motion in the direction of one focal radius vector is equal, or equal and opposite to that in the direction of the other.

264. Roberval's method admits but of limited application in practice, as it is not always easy to see two motions whose ratio can be recognised, and into which the motion of the describing point can be resolved. It is often more convenient to use the following extension, by M. Chasles, of Des Cartes' method (Art. 222) of drawing a tangent to a roulette, and which, though not universally applicable, is so in many important cases: If a plane figure receive any small motion in its plane, there will be always (Art. 246) one point which for the instant remains unmoved, and about which, therefore, all the other points of the figure may be considered as, for the instant, describing circles, and through which, consequently, will pass the normal to the path described by any other point of the figure. Hence if any curve can be considered as described by a point on a moving figure, and if we can perceive the directions of the motion of any two points of the moving figure, the intersection of the normals to their paths will give the instantaneous centre through which the normal to the curve must pass.

Ex. 1. *A line of constant length moves between two directrices; to find the tangent to the path described by any point of it.* Here the extremities of the line describe given curves, the intersection of the normals to which gives the instantaneous centre through which the normal must pass. Thus if a line of constant length move between two right lines, the locus of any point will be an ellipse, the normal to which is constructed by joining the point to the intersection of perpendiculars to the fixed lines at the extremities of the moving line.

Ex. 2. *To construct the tangent to a conchoid, the directrix being any curve.* Here if we consider the right line through the pole as the moving figure, one extremity of it moves on the directrix (a

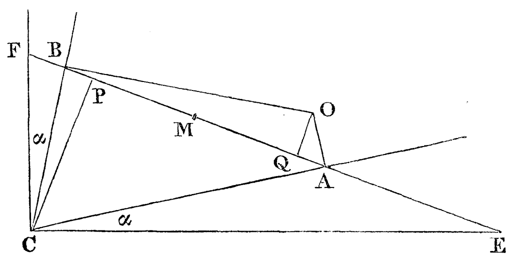
right line for the conchoid of Nicomedes; a circle for the *limaçon*, &c.); the other extremity at the pole moves for the instant along the right line itself. Hence we must erect at the pole a perpendicular to the radius vector to meet the normal to the directrix at the extremity of the radius vector; the line joining their intersection to the describing point will be normal to the curve.

The application of similar methods to the determination of centres of curvature will be found, Liouville, x. 148, 264.

265. If it be required to find the point of contact of a moving figure with its envelope, since the moving figure in any position passes through two consecutive points of the envelope, the path of the point of contact must for the instant coincide with the moving figure, and the point of contact is found by drawing a normal from the instantaneous centre to the moving figure in the corresponding position. If the moving figure be a right line, its point of contact with its envelope will be the foot of the perpendicular on it from the instantaneous centre.

As a first example we give a theorem of M. Chasles. Let it be required *to find the envelope of any diameter of a circle rolling along a right line.* The instantaneous centre being m (see figure, p. 206), the foot of the perpendicular from it on any diameter will lie on a circle described with the diameter cm, and the arc of this circle from m to the point of contact is equal to the corresponding arc of the larger circle; hence the envelope is a cycloid whose generating circle has a radius half that of the given circle. And so in like manner the envelope of any diameter of a circle rolling on another is an epicycloid.

Ex. 2. *To find the point of contact with its envelope, of a right line of constant length between two given right lines.* It must be Q, the foot of the perpendicular from O the instantaneous centre, a point which may otherwise be found by letting fall the perpendicular CP,



and taking BQ = AP (*Conics*, p. 293). The same construction

holds when A and B describe any curves, if AC and BC be the tangents to these curves in any position of AB.

Ex. 3. *To find the evolute of the envelope of* AB *in the last Example.* This is the same as to find the envelope of OQ, which, being normal to the curve, is tangent to the evolute; and we say that the intercept on this line is constant, made by the two bisectors of the angle ACB. For bisect AB at M, and the intercept made on OQ is plainly double the intercept made on the perpendicular to AB at M; but this intercept is the diameter of the circle circumscribing ACB, and this diameter is constant, since we are given the base and vertical angle of the triangle.

266. We may analytically find the equation of the envelope of the second Example. Let us take for axes a pair of rectangular lines equally inclined to the given lines CA, CB; then if $EF = c$, $AB = l$, $PCF = \theta$, we shall find $l = \dfrac{c \sin 2\theta \cos 2\alpha}{\sin 2\theta + \sin 2\alpha}$, and accordingly the equation of the moving line is

$$x \cos\theta + y \sin\theta = \frac{l \sin\theta \cos\theta}{\cos 2\alpha} + \frac{l \sin 2\alpha}{2 \cos 2\alpha}.$$

The equation shows (see p. 97) that if a line of the length $\dfrac{l}{\cos 2\alpha}$ move about in the right angle, ECF, it will be at a constant distance from AB when in the parallel position; and hence it readily follows that the two curves have the same evolute. The equation belongs to the class solved at p. 116, and the envelope in general of

$$x \cos\theta + y \sin\theta = l \sin\theta \cos\theta + d,$$

is $\{x^2 + y^2 - l^2 - \frac{4}{3} d^2\}^3 + 27 \left\{\frac{d}{3}(x^2 + y^2 + 2l^2 - \frac{8}{9} d^2) - lxy\right\}^2 = 0.$

267. We shall next give some examples of the application of the principle already alluded to (*Conics*, p. 209), that since a curve of the n^{th} degree is met by every line in n points: conversely, if we know the number of points in which any line is met by the curve, we shall know the degree of the curve. Of course, in estimating the number of points on any right line, we must be cautious not to neglect any imaginary points; and to take account of all cases where two or more points coincide.

We commence with an important class of theorems, the generalization of Maclaurin's mode of generating conic sections.

Ex. 1. *The three sides of a triangle*, BC, CA, AB, *pass through fixed points*, P, Q, R, *and two of the vertices*, A, B, *move on curves of the* m^{th} *and* n^{th} *degrees respectively; to find the degree of the locus of the third vertex.*

Take any line OP through P, and let us examine in how many points it can meet the locus. Let any one of the m points, m_1, m_2, &c., in which OP meets the curve (M) be joined to R, and let the joining line meet the curve N in the points n_1, n_2, &c., then the line joining any of these to Q will meet OP at a point on the locus. There may be mn such points: the locus is therefore of at least this degree. And it is easy to see that no other point of OP can be a point on the locus, unless the point P itself should be one. But we shall show that the point P is not only on the locus, but also a multiple point of it, of the order mn. For let PQ meet N in N_1, N_2, &c., and let N_1R meet M in M_1, &c., then any of the mn triangles PM_1N_1 is a triangle satisfying the required conditions. The locus is therefore of the $2mn$ degree, and the points P and Q are multiple points of the order mn.

Ex. 2. *In the preceding Example let* P, Q, R *lie on one right line.*

We can prove as before that there are on OP mn points of the locus, distinct from P; but the second part of the proof, which establishes P to be a point on the locus, ceases to be applicable. The locus is therefore only of the degree mn.

Ex. 3. *In the first Example let any of the given points lie on the given curves.*

First, if P lie on the curve M: it appears by the same method as before that there will be on OP $(m-1)\,n$ points of the locus distinct from P: P will be as before a multiple point of the order mn: the degree of the locus will then be $2mn-n$. Q will be a multiple point of the degree $(m-1)\,n$. Secondly, let R lie on either curve, suppose on M. Then the number of points on OP, distinct from P, will be mn, but P will be a multiple point only of the order $n\,(m-1)$; the degree of the locus will then be $2mn-n$, as in the last Example. Q will be a multiple point of the order mn. There is no difficulty in tracing the effect of supposing two

or more of these simplifications of the general problem to take place at once.

Ex. 4. *The three sides of a triangle pass through fixed points, and two vertices move on the same curve of the m^{th} degree; to find the locus of the third vertex.*

By the very same method as before we find for the degree of the locus, $2m(m-1)$; P and Q being both multiple points of the order $m(m-1)$. If both P and Q lie on the curve, the degree of the locus becomes $2(m-1)^2$, for P and Q are multiple points of the order $(m-1)^2$, and any line OP is met by the locus in $(m-1)^2$ points distinct from P.

Ex. 5. *If the sides of a polygon pass through fixed points, and all the vertices but one move on curves respectively of the m^{th}, n^{th}, p^{th}, &c., degrees, to find the locus of the remaining vertex.*

As before, there are on any line OP $mnpqr$, &c., points of the locus distinct from P, and P is itself a multiple point of the degree $mnpqr$, &c., the degree of the locus is therefore $2mnpqr$, &c.

In any Examples of the nature of these here given, when the degree of the locus is once determined, there is no difficulty in constructing it, for we have only to take as many particular positions of the moveable point as are sufficient to determine a curve of the degree of which we find the locus to be.

CHAPTER VII.

APPLICATIONS OF THE INTEGRAL CALCULUS TO CURVES.

268. We have reserved for this Chapter the discussion of those problems concerning curves, the solution of which requires the use of the integral calculus; a subject, perhaps, more necessary to the theoretical completeness than to the practical utility of this treatise; for we must, in this Chapter, suppose the reader already acquainted with the principles of the calculus, and all the works from which he is likely to have derived his knowledge contain illustrations of its application to the theory of curves. We commence by giving the formulæ for the quadrature of

curves. The element of the area, to rectangular co-ordinates, is $dxdy$, which once integrated gives ydx for the element of the area contained between two consecutive ordinates, the curve and the axis of x. If we express y in terms of x, and integrate within any limits, we shall have the area between any two ordinates, the curve and the axis. It is necessary, however, to be careful that y does not change sign within the limits of the integration; otherwise the analytical result will be the difference of two areas, whose separate values if we should wish to know, we must break up the integral into portions, separated by the value of x, for which y changes sign.

In like manner xdy is the element of the area between two abscissæ, the curve and the axis of y. We have obviously

$$\int xdy + \int ydx = xy + c.$$

It was proved (*Conics*, p. 34) that the area of the triangle subtended at the origin by any two points $x'y'$, $x''y''$ is half $x'(y' - y'') - y'(x' - x'')$; if then we suppose these points consecutive, we obtain for the elementary triangle subtended at the origin by the element of the arc, half $xdy - ydx$. Transforming to polar coordinates, we obtain for the elementary triangle, between two consecutive radii vectores and the curve, $\frac{1}{2}\rho^2 d\theta$. This is otherwise obvious, since the base of the triangle is $\rho d\theta$, and its altitude ρ. In integrating this we must, when ρ vanishes or becomes infinite, observe the same caution already noticed in the case of rectangular co-ordinates.

269. Ex. 1. *To find the area of any segment of a circle,*

$$x^2 + y^2 = a^2.$$

The element of the area is $\sqrt{(a^2 - x^2)}\,dx$; and therefore the area

$$= \frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} - \frac{a^2}{2}\cos^{-1}\frac{x}{a} + \text{C}.$$

Taking this between the limits $x = +a$ and $x = -a$, we obtain for the area of the semicircle $\frac{\pi a^2}{2}$; and since, if we give the radical a negative sign, we shall find the area on the negative side of the axis always equal that on the positive side, the entire area of the circle $= \pi a^2$.

Ex. 2. *To find the area of an ellipse,* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The element of the area is $b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}dx$; but this is evidently $\frac{b}{a}$ times the element of the circle described with the radius a. Hence the whole area of the ellipse is to that of the circle in the same ratio.

Ex. 3. *To find the area of an ellipse given by the general equation*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

If we solve for y in terms of x, we get a value of the form $y = P \pm \sqrt{Q}$. But the equation $y = P$ represents the diameter which bisects chords parallel to the axis of y; and $\int Pdx$ represents the area between this diameter and the axis; $\int(P + \sqrt{Q})\,dx$ represents the area between the upper branch of the ellipse and the axis, while $\int(P - \sqrt{Q})\,dx$ represents the area between the lower branch and the axis. It is plain then that $2\sqrt{Q}dx$ is the element of the area of the ellipse itself, and that we shall have the entire area if we use as limits the two values of x for which Q vanishes.

Now we have just seen that $\int_{-a}^{+a}(a^2 - x^2)^{\frac{1}{2}}dx = \frac{\pi a^2}{2}$; but

$$\sqrt{(a + 2bx - cx^2)} = \sqrt{\left(\frac{ac + b^2}{c} - \frac{(cx - b)^2}{c}\right)};$$

and therefore $\int\sqrt{(a + 2bx - cx^2)}\,dx$ between the limits for which the radical vanishes is

$$\frac{\pi(ac + b^2)}{2c^{\frac{3}{2}}}.$$

Expressing a, b, c in the present instance in terms of the coefficients of the equation of the ellipse, we obtain the area

$$= \frac{\pi(AE^2 + CD^2 + FB^2 - ACF - 2BDE)}{(AC - B^2)^{\frac{3}{2}}}.$$

Ex. 4. *To find the area of the curve* $y = Ax^p$.

It is easy to see that when p is greater than -1 the area counted from the origin is $\frac{xy}{p+1}$. When $p = -1$, the curve is the

common hyperbola, and the area depends on logarithms; when p is less than -1, the area of which ydx is the element becomes infinite for $x=0$, but that of which xdy is the element is finite, and proportional to the rectangle under the axes.

Ex. 5. *To find the area of the cissoid of Diocles* (see p. 169).

The entire area, from the cusp to the asymptotes, is readily seen to be equal to three times the area of the generating circle.

Ex. 6. *To find the area of the catenary* (p. 220).

Ex. 7. *To find the area of the tractrix.*

Its differential equation (p. 221) is $ydx = -\sqrt{(h^2 - y^2)}\,dy$; the element of the area is therefore equal to that of a circular segment, and the entire area is equal to that of the circle whose radius is h.

Ex. 8. *To find the area of the cycloid.*

The equations given, p. 207, are

$$y = a(1 + \cos\theta), \quad x = a(\theta + \sin\theta);$$

hence

$$ydx = a^2(1 + \cos\theta)^2 d\theta,$$

$$\int ydx = \tfrac{1}{2}a^2(3\theta + 4\sin\theta + \tfrac{1}{2}\sin 2\theta);$$

and taking this between the limits 0 and 2π, we obtain the entire area $= 3\pi a^2$.

Ex. 9. *To find the area of an epicycloid.*

Its equations are (p. 211)

$$y = b(m\sin\phi + \sin m\phi), \quad x = b(m\cos\phi + \cos m\phi).$$

From the nature of the curve it is most natural to look for the sectorial area described round the centre of the base; this we can do by calculating $xdy - ydx$, which is equal to

$$m(m+1)b^2\{1 + \cos(m-1)\phi\}\,d\phi;$$

the area is therefore

$$\tfrac{1}{2}m(m+1)\,b^2\left(\phi + \frac{1}{m-1}\sin(m-1)\,\phi\right).$$

If we desire to find the area of any one of the similar portions of which (Art. 225) the curve consists, we may use the limits $\phi = 0$, and $\psi = (m-1)\,\phi = 2\pi$; and we find for the area $\dfrac{m(m+1)}{m-1}\pi b^2$. If we wish to calculate the area between this

portion of the epicycloid and the generating circle, we must substract from this the area of the sector of the generating circle, viz., $\frac{1}{2}a^2\phi$ or $(m-1)\pi b^2$; and we obtain for the area required $\frac{3m-1}{m-1}\pi b^2$. The case of the common cycloid is included in this formula by making m infinite.

Ex. 10. *To find the area of an epitrochoid.**

The element of the sectorial area is, as in the last Example,

$$\tfrac{1}{2}m\,(mb^2 + d^2 + (m+1)\,bd\cos(m-1)\,d\phi,$$

the integration of which presents no difficulty when d is less than b, or greater than mb; and the area of any portion is $\frac{m(mb^2+d^2)}{m-1}\pi$; and the area included between the curve and a circle whose radius is $mb-d$ (the minimum value of the radius vector), is $\frac{d\{2mb+(m-1)d\}}{m-1}\pi$. When, however, d is greater than b, and less than mb, the curve has real double points and a series of loops, and accordingly it will happen that the element of the area will change sign when $\cos(m-1)\,\phi = -\frac{mb^2+d^2}{(m+1)}bd$.

In general, in calculating an area by polar co-ordinates, it is plain that every space is reckoned as often as it is traversed by the radius vector, and, therefore, that in the analytical expression for the area the loop is reckoned twice over. If we should desire to find separately the area of the loop, we must find the value of ϕ corresponding to the double point. We cannot apply the ordinary criterion, for in one sense every point of a transcendental epitrochoid is a double point (or at least the double points lie infinitely close on the curve); since any portion of it is crossed by it again at a different point every revolution. But it is obvious that $\phi = 0$, being the value corresponding to the lowest point of the loop, that corresponding to the point which we seek will be the value which next renders $y = mb\sin\phi + d\sin m\phi = 0$ again;

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* Another general method of finding the area of roulettes is given in a paper by Mr. Stubbs, in vol. i. of the Transactions of the Dublin Philosophical Society.

and that if we integrate between these limits we shall have half the area of the loop. The limit cannot be found in finite terms when the curve is transcendental.

Ex. 11. *To find the area of the evolute of an ellipse,* $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$.

The co-ordinates of any point may be expressed

$$x = a\cos^3\phi, \quad y = b\sin^3\phi;$$

and the element of the sectorial area round the origin is, as in the last Examples,

$$\tfrac{3}{2}\,ab\cos^2\phi\sin^2\phi d\phi = \tfrac{3}{8}\,ab\sin^2 2\phi d\phi,$$

the integral of which is

$$\tfrac{3}{16}\,ab\,(\phi - \tfrac{1}{4}\sin 4\phi).$$

If we take this integral between the limits 0 and $\frac{\pi}{2}$, we shall have the area of any one of the four similar portions of which the curve consists, and thus find the entire area $= \frac{3}{8}\pi ab$.

It is easy to see in what cases we can find by this method the area of any curve whose equation is of the form $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

270. We proceed now to give examples of the direct use of polar co-ordinates in calculating areas. Let us commence with the simple case of the circle, the origin being anywhere.

Ex. 1. Its equation is

$$\rho^2 - 2c\rho\cos\theta + c^2 = a^2,$$
$$\rho = c\cos\theta \pm \sqrt{(a^2 - c^2\sin^2\theta)}.$$

Now, first, let the origin be outside the circle, the element of the area of the circle proper is obviously $\frac{1}{2}(\rho_1^2 - \rho_2^2)\,d\theta$, where $\rho_1\rho_2$ are the two values of ρ corresponding to the same value of θ; or is equal to

$$2c\cos\theta\sqrt{(a^2 - c^2\sin^2\theta)}\,d\theta.$$

But if we integrate between the limits $c\sin\theta = \pm a$, we have at once the entire area $= \pi a^2$. Secondly, let the pole be within the circle, we must form $\frac{1}{2}(\rho_1^2 + \rho_2^2)\,d\theta$, and integrate between the limits 0 and π. The element will be $(a^2 + c^2\cos 2\theta)\,d\theta$, and the area, as before, πa^2.

Ex. 2. *To find the area of the conchoid of Nicomedes,*

$$\rho = m \pm p \sec \theta.$$

Ex. 3. *To find the area of the common lemniscata,* $\rho^2 = a^2 \cos 2\theta$.

The entire area $= a^2$.

Ex. 4. *To find the area of the limaçon,* $\rho = a \cos \theta + b$.

There is no difficulty in the integration; when a is less than b the origin is a conjugate point, and we may integrate from 0 to 2π. When a is greater than b, the origin is a double point; and if θ' be the value of θ, for which ρ vanishes, we obtain half the entire area by integrating from 0 to θ', and half the loop by integrating from θ' to π.

Ex. 5. *To find the area of the Cassinoid.*

From the equation

$$\rho^4 - 2a^2\rho^2 \cos 2\omega + a^4 = b^4$$

we have

$$\rho^2 = a^2 \cos 2\omega \pm \sqrt{(b^4 - a^4 \sin^2 2\omega)}.$$

When b is greater than a we must always give the radical the positive sign, and use the limits for ω, 0 and 2π. We should thus obtain the entire area equal to $2b^2$, multiplied by the complete elliptic function of the second kind, whose modulus is $\frac{a^2}{b^2}$.

When b is less than a, the formula $\frac{1}{2}(\rho_1^2 - \rho_2^2)\, d\omega$ gives for the element of the area $\sqrt{(b^4 - a^4 \sin^2 2\omega)}\, d\omega$, which is reduced to elliptic functions by making $a \sin 2\omega = b \sin \phi$. It will be seen from the following Articles that the area in the former case can be expressed geometrically by the help of an arc of an ellipse, and in this case by the help of an arc of a hyperbola.

Ex. 6. *To find the area of the spiral of Archimedes.*

There is no difficulty in the integration for this or any of the other spirals; but the learner must take notice, if he should only want to find the area contained within the outer boundary of the curve after n revolutions, that it will be only necessary to integrate between the limits $(n-1)\pi$ and $n\pi$; since, as we have already observed, in the analytical expression for an area every space is included as many times as it is traversed by the radius vector.

Ex. 7. *To find the area of the involute of the circle.*

Ex. 8. *To find the area of the curve* $x^3 - 3axy + y^3 = 0$.

The area of this curve is found by Bernouilli (see Lacroix, Traité du Calcul Integral, ii. 168) by assuming a new variable z, such that $y = \frac{ax^2}{z^2}$, but a method given by later writers is more convenient, viz., to take as a new variable $\tan\omega = t$; then $y = tx$, $d\omega = \frac{dt}{1+t^2}$; and we have $\rho^2 d\omega = x^2 dt$. The curve has, as may readily be seen, a loop whose area is found by integrating $\frac{1}{2}\left(\frac{3at}{1+t^3}\right)^2 dt$ between the limits 0 and ∞, and is $\frac{3a^2}{2}$.

271. We pass on to give formulæ for the rectification of curves. We have already noticed the formula $ds^2 = dx^2 + dy^2$, which gives the element of the arc for rectangular co-ordinates. By transforming to polar co-ordinates we obtain $ds^2 = d\rho^2 + \rho^2 d\omega^2$, which, however, is otherwise evident, since ds is the hypotenuse of a right-angled triangle, whose sides are $d\rho$ and $\rho d\omega$. We can obtain another formula of great utility when the perpendicular on the tangent p is expressed in terms of the angle θ, which it makes with a fixed axis; for $pd\theta$, the intercept on the tangent between two consecutive perpendiculars, may at once be seen to be the increment of the sum of the tangent (between the foot of the perpendicular and its point of contact) and the arc, measured between the point of contact and any fixed point on the curve.

Ex. 1. *To find the length of the arc of a parabola.*

The arc may be either obtained from the equations

$$2ydy = pdx;\ p^2ds^2 = (p^2 + 4y^2)\,dy^2;$$

or we may use the focal polar equation, and take this curve as a particular example of the class $\rho^n = a^n \cos n\omega$. It was proved (p. 103) that $\rho d\omega = -\cot n\omega d\rho$; hence $ds = \operatorname{cosec} n\omega d\rho$; or, substituting for ρ from the equation of the curve, $ds = -a\cos^{\frac{1}{n}-1} n\omega\, d\omega$, which is always integrable in finite terms when $\frac{1}{n}$ is an integer. In the equation of the parabola

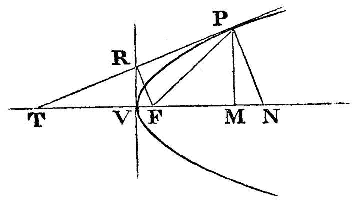
$$\rho = \frac{a}{\cos^2\frac{1}{2}\omega},\ \text{we have}\ n = -\tfrac{1}{2},$$

and
$$ds = \frac{ad\omega}{\cos^3 \frac{1}{2}\omega},$$

$$s = \frac{a \sin \frac{1}{2}\omega}{\cos^2 \frac{1}{2}\omega} + \left(\frac{a}{2}\int \frac{d\omega}{\cos \frac{1}{2}\omega} = \right) a \log \tan\left(\frac{\pi}{4} + \frac{\omega}{4}\right).$$

The algebraic part of the integral is plainly PR, since FV $= a$, and RFV = PFR $= \frac{1}{2}\omega$.

The same expression might then have been derived from the last formula given in this Article, since $pd\theta$ is the element of the difference of the arc PV and PR, but



$$\text{RFV} = \theta, \ \text{FR} = \frac{a}{\cos \theta}.$$

We have seen that the locus of the foot of the perpendicular on the tangent to a curve of the form $\rho^n = a^n \cos n\omega$ is a curve of the same form, the new n being $\frac{n}{n+1}$, the arc of this curve then depends on $\int \cos^{\frac{1}{n}} \omega d\omega$. We see then that if the arc of a curve of this kind depend on $\int \cos^k \omega d\omega$, that of the locus of the feet of perpendiculars on its tangents will depend on $\int \cos^{k+1} \omega d\omega$, and that of the envelope of perpendiculars to its radii vectores at their extremities on $\int \cos^{k-1} \omega d\omega$. If we conceive a series of curves, each being the locus of the feet of perpendiculars from the origin on the tangents to the preceding curve, then the common formula for the reduction of $\int \cos^k \omega d\omega$ gives relations between the arcs of any two alternate curves of the series. (See a memoir by Mr. W. Roberts, Liouville, x. 177.)

272. The relation thus obtained admits also, as Mr. Roberts has remarked, of a simple geometrical expression. First, *the angle which the radius vector makes with the tangent is the same at the corresponding points for all curves of the system.* For the quadrilateral is inscribable in a circle, whose vertices are the origin, a point on one curve, and the feet of perpendiculars on the corresponding tangent and on the consecutive one. But an external angle of this quadrilateral is the angle which the radius vector makes with the line joining two consecutive feet of perpendiculars, and the cor-

responding internal angle is the angle between the radius vector and tangent to the given curve. If therefore ρ, ω belong to the given curve, ρ_1, ω_1 to the locus of feet of perpendiculars, we have $\frac{\rho d\omega}{d\rho} = \frac{\rho_1 d\omega_1}{d\rho_1}$. This is true whatever be the given curve.

Secondly, the element of the arc of the locus is, as we shall prove, $\rho d\omega_1$. For we have just proved $\tan \text{FPR}$ (in the last figure $= \frac{\rho_1 d\omega_1}{d\rho_1}$; but $\text{FR} = \rho_1$, hence $\text{PR} = \frac{d\rho_1}{d\omega_1}$. Substituting this value in the expression $ds_1 = d\omega_1 \sqrt{\left(\rho_1^2 + \frac{d\rho_1^2}{d\omega_1^2}\right)}$, we have, as just stated, $ds_1 = \rho d\omega_1$. Also, if $\rho_{-1}\ \omega_{-1}$ refer to the curve of which the given curve is the locus of the feet of the perpendiculars on the tangent, we have $\rho d\omega$ the element of the sum of the arc s_{-1} and the intercept on the tangent to s_{-1} made by the curve s. Now in the particular class of curves $\rho^n = a^n \cos n\omega$, ω_1 always is in a given ratio to ω (p. 103), and therefore the increment of the arc s_1 is in the same ratio to the increment of the sum of the arc s_{-1} and the tangent; and the quantities themselves are in the same ratio, the arcs being measured from their common summit.

273. Ex. 2. *To find generally when the arc of the curve* $y = \text{A}x^p$ *can be integrated in finite terms.*

The ordinary rules show that the radical on which the integral depends can be rationalized when $\frac{1}{2p-2}$ is an integer, or when $\frac{1}{2p-2} + \frac{1}{2}$ is an integer; or, in other words, when $p =$ either $1 + \frac{1}{2m}$, or $\frac{2m}{2m-1}$, where m may be any integer.

Ex. 3. *To examine when the arc of the curve* $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ *can be found.*

Let us make $x = a \cos^{\frac{2}{m}} \phi$, $y = b \sin^{\frac{2}{m}} \phi$; and putting $\cos 2\phi = x$, it will be readily seen that the rectification of the curve depends on

$$\int dx \sqrt{\{a^2 (1+x)^k + b^2 (1-x)^k\}}, \text{ where } k = \frac{2}{m} - 2.$$

This reduces to known forms when k is an integer. Thus the

element of the arc of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $3a \cos\phi \sin\phi d\phi$, which, integrated between the limits 0 and $\frac{\pi}{2}$, gives $\frac{3a}{2}$ for the length of one of the portions of the curve. To the same class belongs the evolute of the ellipse $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$: only when a curve is known to be the evolute of another, its length is most readily found (see p. 109) by taking the difference of the radii of curvature corresponding to its extremities. To this class belongs $\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}} + \frac{y^{\frac{1}{3}}}{b^{\frac{1}{3}}} = 1$, which (see p. 172) is the most general expression of a parabola of the third degree having a conjugate point. Its arc depends on

$$\int dx \sqrt{\{a^2(1+x)^4 + b^2(1-x)^4\}},$$

and is therefore reducible to elliptic functions. There is no difficulty in finding other curves of the same class whose arcs can be obtained. Thus $x^{\frac{2}{5}} + y^{\frac{2}{5}} = a^{\frac{2}{5}}$ can be rectified by logarithmic functions. The arc of the ellipse itself is of the same class, but, owing to its importance, deserves a closer examination.

274. *To find the length of the arc of an ellipse.*

The substitution just made, viz., $x = a \sin\phi$, $y = b\cos\phi$, gives for the element of the arc $\sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}\, d\phi = a\sqrt{(1 - e^2\sin^2\phi)}\, d\phi$; the arc is therefore expressed by the elliptic function of the second kind. In this expression $\phi = 0$, and the arc is counted to commence, at the extremity of the axis minor.

If we use the formula $pd\theta$, since $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, we get the very same expression for the element of the sum of the arc and tangent, counting from the extremity of the axis major; and thus if we take two points on the curve $x'y'$, $x''y''$, such that the sine of the angle made with the axis of x by the perpendicular on the tangent at $x'y'$ shall $= \frac{x''}{a}$, then the sum of the tangent and arc, measured from the end of the axis major to $x'y'$, will be equal to the arc measured from the extremity of the axis minor to $x''y''$. Similar consequences may be derived from the fundamental formula for the comparison of elliptic functions of the second kind, viz.,

$$E(\phi) + E(\psi) - E(\sigma) = c^2 \sin\phi \sin\psi \sin\sigma,$$

where

$$\cos\sigma = \cos\phi\,\cos\psi - \sin\phi\,\sin\psi\,\sqrt{(1 - c^2\sin^2\sigma)}\,;$$

and by the help of this formula may be proved the same theorems with regard to the possibility of finding an infinity of pairs of arcs whose difference shall be a right line, which have been proved geometrically (*Conics*, p. 296).

275. *To find the length of the arc of a hyperbola.*

The quantity $pd\theta$ will still $= d\theta\sqrt{(a^2 - c^2\sin^2\theta)}$, only that c^2 being $= a^2 + b^2$ is now greater than a^2. This is reduced to elliptic functions by making $c\sin\theta = a\sin\phi$, when it becomes

$$\frac{a\cos^2\phi d\phi}{\sqrt{(e^2 - \sin^2\phi)}} = ae\sqrt{\left(1 - \frac{1}{e^2}\sin^2\phi\right)} - \frac{ae\left(1 - \frac{1}{e^2}\right)}{\sqrt{\left(1 - \frac{1}{e^2}\sin^2\phi\right)}}.$$

The difference then between the arc and tangent is expressed by means of an elliptic function of the first and one of the second species having the common modulus $\frac{1}{e}$. Taking the complete functions, we get the difference between the asymptote and the infinite hyperbolic arc.

The formulæ already alluded to for comparing three functions having a common modulus enable us, in an infinity of ways, to find two hyperbolic arcs whose difference shall be an algebraic quantity.

Landen proved, in 1780, that an arc of the hyperbola can be expressed by two elliptic arcs. This is an immediate consequence of Lagrange's formula of reduction (Legendre's Exercises, i. 85),

$$b^2F(c,\phi) = 2E(c,\phi) - (2 + 2c)\,E(c_1, \phi_1) + 2c\sin\phi,$$

where

$$c = \frac{1 - b_1}{1 + b_1}, \quad \tan(\phi - \phi_1) = b_1\tan\phi_1.$$

The reader will find a geometrical demonstration of the same theorem by Mr. Mac Cullagh, Transactions of the Royal Irish Academy, vol. xvi. p. 80.

276. *To find the length of the arc of a common lemniscata,*

$$\rho^2 = a^2\cos 2\theta.$$

This is a particular case of the formula of Art. 271, and gives $ds = \dfrac{a d\theta}{\sqrt{(\cos 2\theta)}}$, which, putting $2\sin^2\theta = \sin^2\phi$, becomes

$$\frac{a}{\sqrt{2}} \frac{d\phi}{\sqrt{(1 - \frac{1}{2}\sin^2\phi)}}.$$

The arc is then expressed by an elliptic function of the first kind whose modulus is $\sqrt{\frac{1}{2}}$. The same result is found by Legendre by expressing the co-ordinates as a function of the radius vector,

$$x = \frac{\rho}{a}\sqrt{\left(\frac{a^2+\rho^2}{2}\right)}, \quad y = \frac{\rho}{a}\sqrt{\left(\frac{a^2-\rho^2}{2}\right)}, \quad ds = \frac{-a^2 d\rho}{\sqrt{(a^4-\rho^4)}},$$

which is reduced to the preceding form by making $\rho = a\cos\phi$. Or M. Serret has expressed x and y rationally in terms of the amplitude of this function

$$x = a\sqrt{2}\frac{z+z^3}{1+z^4}, \quad y = a\sqrt{2}\,\frac{z-z^3}{1+z^4}, \quad ds = 2a\,\frac{dz}{\sqrt{(1+z^4)}}.$$

In consequence of the arcs of the lemniscata being expressible by elliptic functions of the first kind, they can, by the fundamental property of these functions, be added, subtracted, multiplied, or divided algebraically, just like arcs of circles.

277. The lemniscata is, as we know, the locus of the feet of perpendiculars on the tangents of an equilateral hyperbola. The difference between the asymptote and the infinite hyperbolic arc is given by the equation (see Art. 275)

$$S = \frac{a}{\sqrt{2}}\left\{2E\left(\frac{1}{\sqrt{2}}\right) - F\left(\frac{1}{\sqrt{2}}\right)\right\},$$

and we have seen that the complete quadrant of the lemniscata is

$$S_1 = \frac{a}{\sqrt{2}}F\left(\frac{1}{\sqrt{2}}\right).$$

Now Legendre's formula for complete functions with complementary moduli,

$$E(c)\,F(b) + E(b)\,F(c) - F(c)\,F(b) = \frac{\pi}{2},$$

becomes, when $c = b = \dfrac{1}{\sqrt{2}}$,

$$F\left(\frac{1}{\sqrt{2}}\right)\left\{2E\left(\frac{1}{\sqrt{2}}\right) - F\frac{1}{\sqrt{2}}\right\} = \frac{\pi}{2};$$

and hence

$$SS_1 = \frac{\pi a^2}{4}.*$$

278. We may form the curve, which is the locus of the feet of perpendiculars on the tangents to a lemniscata, and may conceive another curve generated from it in like manner, and so on. Mr. Roberts has proved (Liouville, x. 177), and it follows readily from what we have said (Art. 271), that if we obtain by the formulæ of reduction the relation between the arcs of two non-consecutive curves of the series, and by taking complete quadrants make the algebraic part of the formula to vanish, we shall have

$$\frac{S_{n-1}}{S_{n+1}} = \frac{2n-1}{2n+1}.$$

The quadrants of all the even curves of the series being expressed by S of the last Article, and those of the odd curves of the series by S_1, he obtains easily

$$S_n\,S_{n+1} = \frac{2n+1}{4}\,\pi a^2.$$

279. The remarkable property of the lemniscata, that its arcs can be added, multiplied, or divided algebraically, led geometers to inquire whether there were not other curves whose arcs, being expressed by elliptic functions of the first kind, enjoyed the same property. They sought then to find curves whose arcs should represent elliptic functions of the first kind in general, which the lemniscata only does for the particular case of $c^2 = \frac{1}{2}$. This is one of a class of problems, the inverse of rectification. In the latter it is required to find the integral corresponding to the arcs of a given curve, in the present instance to find the curve whose arcs are expressed by a given integral. If in this Example it be not required that the curve shall be algebraical, the problem presents no difficulty. The following is Legendre's solution (Exercises, i. 39):

* This theorem is due to Mr. Talbot (see Gergonne, xiv. 17), but the proof here given is due to Mr W. Roberts, who has also extended the theorem to the case of any hyperbola, and the locus of the feet of perpendiculars from the centre on its tangents. (Liouvilla, xii. 41, xiii. 179.) For a generalization, by Mr. Roberts, of the property of the next Article, when the original curve is any of the form $\rho^m = a^m \cos m\omega$, see Liouville, xv. 213.

$$dx = \frac{d\phi\,(\cos\phi\cos\psi - b\sin\phi\sin\psi)}{1 - c^2\sin^2\phi},$$

$$dy = -\frac{d\phi\,(\cos\phi\sin\psi + b\sin\phi\cos\psi)}{1 - c^2\sin^2\phi},$$

whence
$$ds = \frac{d\phi^2}{1 - c^2\sin^2\phi}.$$

We may take $\sin\psi$ at pleasure any function of ϕ, and integrate if we can. For example, for $\psi = 0$ we have

$$x = \frac{1}{2c}\log\left(\frac{1 + c\sin\phi}{1 - c\sin\phi}\right), \quad y = \frac{1}{c}\tan^{-1}\left(\frac{c\cos\phi}{b}\right),$$

from which two equations the curve is determined.

280. When, however, it is required that the curve shall be algebraical the problem is one of much greater difficulty. If, indeed, it would suffice that the arcs of the sought curve should only differ by an algebraical quantity from elliptic functions of the first kind, the fact that $pd\theta$ is the increment of the sum of the arc and tangent would present us with an immediate solution. Thus in this Example it is only necessary to find a curve, the perpendicular on the tangent to which shall be proportional to $\dfrac{1}{\sqrt{(1 - c^2\sin^2\theta)}}$; or, in other words, to find the curve which is the envelope of the perpendiculars to the diameters of an ellipse erected at their extremities. The equation of one of these perpendiculars is

$$xx' + yy' = x'^2 + y'^2,$$

$x'y'$ being the co-ordinates of the extremity of the diameter; or substituting $a\cos\phi$, $b\sin\phi$, for $x'y'$,

$$ax\cos\phi + by\sin\phi = a^2\cos^2\phi + b^2\sin^2\phi,$$

an equation of the class discussed, p. 116, and whose envelope therefore is

$$\{4\,(a^4 + b^4 - a^2b^2) - 3\,(a^2x^2 + b^2y^2)\}^3$$
$$= \{9a^2(2b^2 - a^2)\,x^2 + 9b^2(2c^2 - b^2)\,y^2 - 4\,(a^2 + b^2)\,(2a^2 - b^2)\,(2b^2 - a^2)\}^2.*$$

* This example illustrates well the advantage gained by the form into which Messrs. Boole and Cayley (see Appendix) have thrown the envelope of an equation involving a

The arcs of this curve then will be expressed by an elliptic function of the first kind (whose modulus is the excentricity of the generating ellipse), together with an algebraical part; and integrating between 0 and $\frac{\pi}{2}$, the algebraic part vanishes, and the quadrant of the curve expresses the complete function F without any addition. In consequence, however, of the algebraic part in the general case, we cannot find a given submultiple of a given arc of this curve, but only an arc whose length shall differ from that submultiple by an algebraical quantity. This is only what we can do for the ellipse, but comes much short of the simplicity of the lemniscata.

281. It was M. Serret who first gave a general solution of the problem to find the real and rational solutions which the equation

$$dx^2 + dy^2 = Zdz^2$$

admits of, and who proved in particular that there are an infinity of algebraic curves whose arcs can be expressed by elliptic functions of the first kind. Referring to his original memoirs (Liouville, x. 257, 351, 421) for details as to his general method, we shall substitute here a more elementary investigation, which, however, will suffice to include the most interesting of his results. However fortunate it was for the interests of analysis that M. Ser-

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variable parameter in the fourth degree. The elimination necessary for the solution of the present problem evidently baffled Legendre, and some other succeeding writers on the subject, and was only performed not very long ago by Tortolini (Crelle, vol. xxxiii. p. 90, "Nota sopra l'equazione di una curva del sesto ordine"), after a very creditable degree of skilful labour. It is evident that the method used here would equally show that the equation of the envelope, in general, of perpendiculars at the extremities of radii vectores to a conic *from any origin*, can be exhibited under the form $S^2 = T^3$. For the problem is equivalent to finding the reciprocal of the inverse of the ellipse, which latter, being of the fourth degree, its reciprocal is found by Art. 109. The most interesting case is where the origin is the focus, when the problem is reduced to finding the reciprocal of a limaçon, and depends on an equation of the form

$$Ax \cos 2\theta + Ay \sin 2\theta + 4Bx \cos\theta + 4By \sin\theta + 3C = 0,$$

where A, B are constants, and C contains x in the first degree. The envelope is

$$\{\rho^2 (A^2 - 4B^2) + 3C^2\}^3 = 27\{(A^2 + 2B^2) C\rho^2 - 2AB^2 x\rho^2 - C^3\}^2,$$

an equation obviously divisible by ρ^2, and thus reduced to the fourth degree.

ret should have attacked and conquered the problem in its most difficult form, we may wonder that Legendre and the other mathematicians who had attempted the solution of the question, had not observed how much it is simplified by the use of polar coordinates. Thus if, according to the most natural assumption, we take

$$dx^2 + dy^2 = \frac{dx^2}{X},$$

where X is a complete algebraic function of the fourth degree in x, and if we solve for dy, we get an equation not algebraically integrable; but if we take

$$d\rho^2 + \rho^2 d\theta^2 = \frac{d\rho^2}{R},$$

where R is a similar function of ρ; then, if we solve for $\rho^2 d\theta^2$, if the numerator be a perfect square, and if R only involve even powers of ρ, the substitution of $\rho^2 = z$ will give an equation for $d\theta$ integrable by circular functions. We are thus naturally led to the assumption

$$d\rho^2 + \rho^2 d\theta^2 = \frac{a^4 d\rho^2}{a^4 - (\rho^2 - b^2)^2},$$

whence

$$d\theta = \frac{(\rho^2 - b^2)\, d\rho}{\rho \sqrt{\{a^4 - (\rho^2 - b^2)^2\}}},$$

$$2\theta = \sin^{-1}\left(\frac{\rho^2 - b^2}{a^2}\right) + \frac{b^2}{\sqrt{(b^4 - a^4)}} \sin^{-1} \frac{b^4 - a^4 - b^2\rho^2}{a^2\rho^2}.$$

If then we take any rational number $= \frac{b^2}{\sqrt{(b^4 - a^4)}}$, we can at once form the polar equation of the curve, which will in this case be always algebraical.

282. These coincide with M. Serret's first class of elliptic curves. Of these he gives the following geometrical definition:— With the radius vector OP as base, let a triangle be constructed whose sides OM, MP are proportional to $\sqrt{n}$, $\sqrt{(n+1)}$, (n being any rational number), and if the angles opposite these sides be β, α, then we shall have

$$\theta = n\alpha - (n+1)\beta.$$

For we have, writing for brevity,

$$\Delta = \sqrt{\{-\rho^4 + 2(2n+1)\rho^2 - 1\}},$$

$$\cos\alpha = \frac{\rho^2 - 1}{2\rho\sqrt{n}}, \quad \sin\alpha = \frac{\Delta}{2\rho\sqrt{n}};$$

$$\cos\beta = \frac{\rho^2 + 1}{2\rho\sqrt{(n+1)}}, \quad \sin\beta = \frac{\Delta}{2\rho\sqrt{(n+1)}};$$

$$d\alpha = -\frac{(\rho^2+1)}{\Delta}\frac{d\rho}{\rho}; \quad d\beta = -\frac{\rho^2-1}{\Delta}\frac{d\rho}{\rho};$$

$$d\theta = n\,d\alpha - (n+1)\,d\beta = \frac{\rho^2 - (2n+1)}{\Delta}\frac{d\rho}{\rho};$$

and hence

$$ds = 2\sqrt{\{n(n+1)\}}\,\frac{d\rho}{\Delta}.$$

These can be readily made to coincide with the form given in the last Article; for though M. Serret's form appears to contain a constant less, it is only because one of the constants is for simplicity made unity, which only affects the size, not the shape of the curve.

The expression for the arc is reduced to the ordinary form by taking

$$\sin\beta = \sqrt{\left(\frac{n}{n+1}\right)}\sin\alpha = k\sin\alpha,$$

when we have

$$ds = \sqrt{n}\frac{d\alpha}{\sqrt{(1 - k^2\sin^2\alpha)}};$$

or the arc is expressed by an elliptic function of the first kind, whose modulus is k.

283. The sectorial area of these curves is readily found to be always $\frac{1}{4}\Delta$; or the area of the sector is equal to that of the generating triangle. The tangent is constructed by joining the point on the curve to the centre of the circle circumscribing the generating triangle. M. Serret has proved that when n is a fraction $\frac{p}{q}$, the degree of the curve is $2(p+q)$, and consequently that two fractions $\frac{p}{q}, \frac{q}{p}$ give curves of the same degree whose arcs depend on functions with complementary moduli.

The following are the equations of the simplest of these curves. First, let $n = 1$; we have

$$\cos\theta = \cos(\alpha - 2\beta) = \frac{\rho^4 + 4\rho^2 - 1}{4\rho^3},$$

or

$$\rho^2(\rho^2 - 4\rho\cos\theta + 4) = 1,$$

the equation of a lemniscata, the distance between the foci being 2, and the product of the focal radii $= 1$.

Let $n = \frac{1}{2}$ and $\cos 2\theta = \cos(\alpha - 3\beta) = \dfrac{\rho^6 + 6\rho^2 - 2}{3\sqrt{3}\rho^4}$.

Let $n = 2$ and $\cos\theta = \cos(2\alpha - 3\beta) = \dfrac{4\rho^6 + 27\rho^4 - 12\rho^2 + 1}{12\sqrt{3}\rho^5}$.

These two curves of the sixth degree whose arcs depend on functions whose moduli are $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{2}{3}}$, are the simplest of M. Serret's next to the lemniscata.

284. Since the publication of M. Serret's memoirs, Mr. Roberts has found another couple of very simple curves, whose arcs are expressible by elliptic functions of the first kind. (See Liouville, xii. 445.) If the arc of the curve, $\rho^m = a^m \cos m\omega$, be expressed as a function of the radius vector, we shall have in general

$$ds = \frac{a^m d\rho}{\sqrt{(a^{2m} - \rho^{2m})}}.$$

But when $m = 3$, this is at once reducible to elliptic functions by the substitution $\rho^2 = z$; and hence it appears that the arcs of the curve, $\rho^3 = a^3 \cos 3\omega$, are expressed by an elliptic function of the first kind, whose modulus is $\sin\dfrac{\pi}{12}$. They may equally be expressed by the function with the complementary modulus, by the help of the relation given by Legendre (Elliptic Functions, vol. i. p. 185) for connecting these two functions. This curve is also in another respect analogous to the lemniscata; for the locus of a point, the product of whose distances from the vertices of an equilateral triangle is constant, is

$$\rho^6 - 2a^3\rho^3 \cos 3\omega + a^6 = b^6,$$

and becomes of the form in question when $a = b$.

The general formula given above proves that the arcs of the

curve $\rho^{\frac{3}{2}} = a^{\frac{3}{2}} \cos \frac{3}{2}\omega$ are similarly expressed by elliptic functions of the first kind.

285. *To find the length of the arc of the cissoid of Diocles.*

The arc is found by elementary integration, and depends on logarithmic functions; the arc, in general, of a circular cubic with a double point depends on elliptic functions.

To find the length of an arc of an epicycloid or epitrochoid.

Using the notation of p. 211, we at once have

$$ds = m\sqrt{\{b^2 + d^2 - 2bd\cos(m-1)\phi\}}\,d\phi,$$

which enables us to express the arc of an epitrochoid by means of the arc of an ellipse. If $b = d$ the curve is an epicycloid, and the length of one of its portions is $\frac{8m}{m-1}\,b$, which, by making m infinite, gives the length of the cycloid.

To find the length of an arc of the tractrix.

The differential equation (p. 221) gives at once $yds = hdy$.

To find the length of the involute of the circle.

286. *To find the length of an arc of the oval of Cassini.*

The equation being

$$\rho^4 - 2a^2\rho^2\cos 2\omega + a^4 = b^4,$$

we have

$$(\rho^2 - a^2\cos 2\omega)\,d\rho + a^2\rho\sin 2\omega d\omega = 0,$$

$$ds = \frac{b^2 d\rho}{a^2\sin 2\omega} = \frac{2b^2\rho^2 d\rho}{\sqrt{\{-\rho^8 - 2(a^4+b^4)\rho^4 - (a^4-b^4)^2\}}}.$$

But Legendre has proved (Exercises, i. 197) that

$$\int \frac{x^2dx}{\sqrt{(\alpha + \beta x^4 + \gamma x^8)}}$$

can be reduced to the sum of two functions of the first kind with complementary moduli, by first reducing the denominator to a recurring function by the substitution $\gamma x^8 = \alpha y^8$, and then making $y + \frac{1}{y} = z$.

For the general class of curves,

$$\rho^{2m} - 2a^m\rho^m\cos m\omega + a^{2m} = b^{2m},$$

M. Serret has proved in like manner that the differential of the arc is

$$ds = -\frac{2b^m\rho^m d\rho}{\sqrt{\{-\rho^{4m} + 2(a^{2m} + b^{2m})\rho^{2m} - (a^{2m} - b^{2m})^2\}}},$$

which Mr. Roberts has showed can be reduced by the preceding substitutions to the sum of two Abelian functions of lower dimensions. (See Liouville, viii. 145, 501; xiii. 38.)

287. *To find the length of the arc of a Cartesian oval.*

The expression for the arc itself depends on ultra-elliptic functions; but Mr. Roberts (Liouville, xv. 195), extending a method applied by M. Serret to the last example, has proved that the difference between the two arcs of the conjugate ovals, corresponding to the same position of the radius vector, can be expressed by an arc of an ellipse. In general, let the equation of a curve be

$$\rho^2 - 2\rho\Omega + C = 0,$$

where C is constant, and Ω any function of ω, then we have at once

$$\rho = \Omega \pm \sqrt{(\Omega^2 - C)}, \quad (\rho - \Omega)\, d\rho = \Omega'\rho d\omega,$$

$$ds = \frac{\sqrt{(\Omega^2 + \Omega'^2 - C)}}{\sqrt{(\Omega^2 - C)}}(\Omega \pm \sqrt{(\Omega^2 - C)}\, d\omega.$$

The element, then, of the difference of two corresponding arcs is

$$2\sqrt{(\Omega^2 + \Omega'^2 - C)}\, d\omega.$$

Applying this to the Cartesian oval, whose equation is of the form

$$\rho^2 - 2\rho(a + b\cos\omega) + c^2 = 0,$$

the element of the difference in question is

$$2\sqrt{(a^2 + 2ab\cos\omega + b^2 - c^2)}\, d\omega,$$

which obviously represents an elliptic arc.

288. We shall next give some formulæ connecting the areas and arcs of different curves. If from the equation of any curve to rectangular co-ordinates we form an equation to polar co-ordinates by taking $dy = d\rho$, $dx = \rho d\omega$, then it is obvious that the element of the arc of one curve will be equal to that of the other; the element of the rectangular area of the first will be double that of the sectorial area of the second; and the second curve may be conceived as generated from the first by bending all its ordinates into radii vectores. Thus from the parabola $2ydy = pdx$ we get

$2\rho d\rho = p\rho d\omega$, $2\rho = p\omega$; or the rectification and quadrature of the spiral of Archimedes depend on the same formulæ as in the case of the parabola.*

The following is a method by which rectification may in general be reduced to quadratures, and by which, in particular, Van Huraet rectified the semicubical parabola, the first curve rectified. (Lardner's Algebraic Geometry, p. 464.) Produce each ordinate until the whole length be in a constant ratio to the corresponding normal divided by the old ordinate, and the locus of the extremity of the produced ordinate is a curve whose area is in a constant ratio to the length of the given curve. The rectification of the parabola is thus reduced to the quadrature of the hyperbola, and the rectification of the semicubical parabola to the quadrature of the parabola. The truth of the theorem is obvious.

The reader will find general formulæ by Mr. Roberts connecting the element of the length of a curve with that of the locus of the feet of perpendiculars on its tangents (Liouville, x. 177). The most interesting application of these is to the case of the ellipse considered as the curve, the locus of feet of perpendiculars on whose tangents is a circle; when the same expression is found for the length of its arc as that obtained by Lagrange's modular transformation. It is an interesting subject of inquiry whether there may not be also some geometrical point of view which would lead directly to the more general transformations of Jacobi and Abel.

289. When a curve is transformed by the method of Art. 260† it is easy to find an expression for the arc of the transformed curve. Let this be σ, that of the old curve s, and let ρ be the old radius vector, then we have at once,

$$d\sigma = \frac{ds}{n\rho^{\frac{n-1}{n}}}.$$

* This method is due to Gregory of St. Vincent.—See Leslie's Geometrical Analysis, p. 424; Lardner's Algebraic Geometry, pp. 356, 494.

† I find that I was mistaken in supposing (see p. 239) that M. Chasles had published anything on this subject beyond the note to the Aperçu Historique already referred to; and Mr. Roberts tells me that he was not acquainted with this note at the time when, on M. Liouville publishing some discussions on the transformations by reciprocal radii vectores (vol. xii. p. 256), he wrote to inform him that this method had already been given by Mr. Stubbs, giving at the same time his own more general method.

This formula is applied by Mr. Roberts to the rectification of the class of curves which are the generalization of Cassini's ovals,

$$\rho^{2n} - 2a^n\rho^n \cos n\omega + a^{2n} = b^{2n}.$$

We have already seen that since the circle

$$\rho^2 - 2a\rho \cos\omega + a^2 = b^2$$

is always cut orthogonally by the system of right lines,

$$\rho \cos(\omega - \theta) = a \cos\theta,$$

the curve in question will be always cut orthogonally by the system of curves,

$$\rho^n \cos n(\omega - \theta) = a^n \cos n\theta.$$

But considering θ as the variable parameter, the arc of the circle is obviously $bd\theta$, and the two values of ρ corresponding to any value of θ are readily seen to be given by the equation

$$\rho^2 = a^2 \pm 2ab \sin\theta + b^2;$$

hence by the formula of this Article, the element of the required arc is

$$\frac{b^n d\theta}{(a^{2n} \pm 2a^n b^n \sin n\theta + b^{2n})^{\frac{n-1}{2n}}}.$$

290. We proceed now to other problems for which the integral calculus is necessary. We pass over a class of problems where it is required to find the curve, being given a relation between two lines connected with it; for example, "to find the curve such that the normal shall be always equal to the radius vector;" there being in general no difficulty in finding the differential equation, on the solution of which the problem depends. We pass on to the problem of trajectories, where it is required to find a curve cutting a given system of curves at a given angle.

Let the system of given curves be $F(\xi, \eta, c) = 0$, where c is a variable parameter, then we have the condition

$$\tan\phi\left(1 + \frac{dy}{dx}\frac{d\eta}{d\xi}\right) = \frac{dy}{dx} - \frac{d\eta}{d\xi},$$

where ϕ is the given angle. If in this equation we substitute x and y for ξ and η in the value of $\frac{d\eta}{d\xi}$, obtained from the equation

of the given curve, we shall have a relation which must be satisfied for the point where any curve is cut by the trajectory; and if we eliminate c between this equation and the given one, we shall have the relation satisfied by all the points of the trajectory. For orthogonal trajectories the equation reduces to

$$1 + \frac{dy}{dx}\frac{d\eta}{d\xi} = 0.$$

Ex. To find the trajectory cutting at a given angle the series of curves,

$$\eta^n = c\xi^m,$$

we have then

$$\frac{d\eta}{d\xi} = c\frac{m}{n}\frac{x^{m-1}}{y^{n-1}} = \frac{my}{nx};$$

and the differential equation of the trajectory is

$$\tan\phi\,(nxdx + mydy) = nxdy - mydx,$$

which, being homogeneous, the variables can be separated by the substitution $y = xz$. For the treatment of cases where the elimination of c is more difficult, see Lacroix's Integral Calculus, ii. 453; Euler, Novi Comm. Acad. Petrop., vol. xiv. part i. p. 26; xvii. p. 205.

One case, however, it is necessary to mention. It is when one variable is given as a function of the other and the parameter, and when it is required to find the orthogonal trajectory. Thus let $\eta = \phi(\xi, c)$; $d\eta = pd\xi$; then the equation of the orthogonal trajectory is, as before,

$$dx + pdy = 0,$$

and it is necessary to introduce in p the value of c derived from the equation $y = \phi(x, c)$. But it is more convenient in practice to bring this to a differential equation between x and c. If by differentiating the equation, considering c variable, we have

$$dy = pdx + qdc,$$

then substituting this value of dy in the equation of the trajectory, we shall have

$$(1 + p^2)\,dx + pqdc = 0,$$

an equation not involving y, which integrated gives a relation between x and c, from which relation and the equation $y = \phi(x, c)$, c can be algebraically eliminated.

291. The problem to find the involute of a given curve is a particular case of the problem of orthogonal trajectories, since it is required to find the curve cutting at right angles all the tangents to the given curve. Thus (see Gregory's Examples, p. 448) if the curve be $\eta = \phi(\xi)$, $d\eta = pd\xi$, then the equation of any tangent is

$$(y - \eta) = p(x - \xi),$$

where η, p are known functions of ξ. This, then, is a particular case of the last; and if we differentiate, considering ξ variable, we shall have

$$dy = pdx + (x - \xi)\,dp.$$

Hence for the equation of the trajectory we have

$$(1 + p^2)\,dx + (x - \xi)\,pdp,$$

which can be integrated when from the equation of the given curve we express ξ in terms of p. It is reduced to the most convenient form by dividing by $\sqrt{(1 + p^2)}$. Then

$$\sqrt{(1 + p^2)}\,dx + \frac{xpdp}{\sqrt{(1 + p^2)}} = \frac{\xi pdp}{\sqrt{(1 + p^2)}},$$

$$x\sqrt{(1 + p^2)} = \xi\sqrt{(1 + p^2)} - \int\sqrt{(1 + p^2)}\,d\xi.$$

When this integration can be accomplished we have x in terms of p or of ξ, which we can then eliminate between this equation and that of the tangent.

Ex. *To find the involute of the semicubical parabola,*

$$27a\eta^2 = 4\xi^3.$$

Then
$$p^2 = \frac{\xi}{3a};\ \int\sqrt{(1 + p^2)}\,d\xi = 2a\left(1 + \frac{\xi}{3a}\right)^{\frac{3}{2}} + C;$$

and we have

$$x\left(1 + \frac{\xi}{3a}\right)^{\frac{1}{2}} = \xi\left(1 + \frac{\xi}{3a}\right)^{\frac{1}{2}} - 2a\left(1 + \frac{\xi}{3a}\right)^{\frac{3}{2}} - C.$$

The particular involute depends on the constant C, and the simplest is found by making it $= 0$; when we have

$$x = \xi - 2a\left(1 + \frac{\xi}{3a}\right) = \frac{\xi}{3} - 2a,$$

and by the equation of the tangent

$$y - 2ap^3 = p(x - \xi),$$

we have
$$y^2 = 4a^2p^2 = \tfrac{4}{3}a\xi = 4a(x + 2a),$$

the equation of a parabola.

292. Similarly, by giving C any value, we could find all the curves whose evolute is a given semicubical parabola. The problem, however, is simpler when one involute of a curve is given, and it is required to find the rest. It is obvious from Art. 115 that this is equivalent to finding the locus of points whose normal distance from the given involute shall be constant, or to finding the envelope of lines parallel to the tangents of the curve, and at a constant distance from them. Hence if we substitute in the equation of the polar reciprocal of the given involute $\frac{1}{\rho}+d$ for $\frac{1}{\rho}$, we shall have the reciprocal of the required curve. It easily appears thus that the problem to find the curve which shall have the same involute as a given conic is reduced to finding, by Art. 109, the reciprocal of a curve of the fourth degree. But we may also solve the question directly. Thus let it be required *to find the curve which shall have the same evolute as the parabola*

$$x^2 + y^2 = (x + 2m)^2.$$

The equation of any tangent (*Conics*, p. 185) is

$$x \cos\theta + y \sin\theta + \frac{m}{\cos\theta} = 0.$$

Hence that of a parallel at a constant distance is

$$x \cos\theta + y \sin\theta + \frac{m}{\cos\theta} + 4d = 0,$$

or $$x\cos 2\theta + y\sin 2\theta + 4d\cos\theta + (x + 2m) = 0,$$

whose envelope is

$$\{3(x^2 + y^2) - 12d^2 + (x + 2m)^2\}^3$$
$$= \{9(x + 2m)(x^2 + y^2 + 2d^2) - (x + 2m)^3 - 54xd^2\}^2,$$

which is readily seen to be of the form

$$27(x^2 + y^2)\{(x^2 + y^2) - (x + 2m)^2\}^2 + d^2S = 0.$$

293. Similarly *we can find the curves which have the same evolute as the ellipse*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equation of any tangent to the ellipse being

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0,$$

that of any tangent to the required curve will be

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = d\sqrt{\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)}.$$

This is reduced to the ordinary form by squaring; and the envelope is $4S^3 = T^2$, where

$$S = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 + \frac{d^2}{a^2b^2}\left\{(a^2 - 2b^2)\frac{x^2}{a^2} + (b^2 - 2a^2)\frac{y^2}{b^2} - (a^2 + b^2)\right\}$$
$$+ \frac{d^4}{a^4b^4}(a^4 - a^2b^2 + b^4);$$

$$T = 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^3$$
$$+ \frac{3d^2}{a^2b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left\{(a^2 - 2b^2)\frac{x^2}{a^2} + (b^2 - 2a^2)\frac{y^2}{b^2} - (a^2 + b^2)\right\}$$
$$+ \frac{3d^4}{a^4b^4}\left\{(2b^4 - 2a^2b^2 - a^4)\frac{x^2}{a^2} + (2a^4 - 2a^2b^2 - b^4)\frac{y^2}{b^2} + (a^4 - 4a^2b^2 + b^4)\right\}$$
$$- \frac{d^6}{a^6b^6}(2a^6 - 3a^4b^2 - 3a^2b^4 + 2b^6).$$

When the equation is developed, the terms above the eighth degree in x and y vanish identically, and the equation becomes divisible by d^4; and it will be found that the result of then making $d = 0$ in it will be

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 \{y^2 + (x - c)^2\} \{y^2 + (x + c)^2\} = 0.$$

It is not difficult to see the geometrical reason why the two foci thus appear as points whose normal distance from the curve vanishes.

294. We add a notice of some cases in which the orthogonal trajectory may be found more expeditiously than by the general method.

Let it be required to find the orthogonal trajectory of curves given by the equation

$$\rho_1{}^l \rho_2{}^m \rho_3{}^n \text{ \&c.} = k^{l+m+n+,\text{ \&c.}}$$

where ρ_1, &c. denote the distance of a variable point from given

fixed points; l, m, &c. are given constants, and k the variable parameter; then for any of these curves we have

$$l\frac{d\rho_1}{\rho_1} + m\frac{d\rho_2}{\rho_2} + \&c. = 0;$$

or if i_1 be the angle which the tangent makes with the radius vector ρ_1, since $d\rho_1 = ds \cos i_1$,

$$l\frac{\cos i_1}{\rho_1} + m\frac{\cos i_2}{\rho_2} + \&c. = 0;$$

hence for the orthogonal trajectory,

$$l\frac{\sin i_1}{\rho_1} + m\frac{\sin i_2}{\rho_2} + \&c. = 0;$$

but since $\rho_1 d\omega_1 = ds \sin i_1$,

$$l d\omega_1 + m d\omega_2 + \&c. = 0,$$

$$l\omega_1 + m\omega_2 + \&c. = C,$$

which condition determines the orthogonal trajectory.

Ex. 2. *To find the orthogonal trajectory of the confocal Cartesian ovals given by the equation*

$$l\rho_1 + m\rho_2 = c,$$

where l, m are constant, c a variable parameter.

For the Cartesian oval we have (p. 176)

$$l\cos i_1 + m\cos i_2 = 0;$$

hence for the trajectory,

$$l\sin i_1 + m\sin i_2 = 0; \quad l\rho_1 d\omega_1 + m\rho_2 d\omega_2 = 0.$$

But ω_1 and ω_2 being taken for the base angles of the triangle, whose sides are ρ_1, ρ_2, it follows that

$$\frac{l d\omega_1}{\sin\omega_1} \pm \frac{m d\omega_2}{\sin\omega_2} = 0;$$

and hence

$$\tan^l \tfrac{1}{2}\omega_1 = C\tan^{\pm m}\tfrac{1}{2}\omega_2,$$

a relation which determines the trajectory. When $l = m$ the Cartesian oval becomes a central conic, and the trajectory is the locus of the vertex of a triangle whose base and ratio or product of

tangents of half base angles is given; that is, an hyperbola or ellipse.*

Ex. 3. Let the curves be given by the relation

$$p_1{}^m p_2{}^n = c^{m+n},$$

where $p_1 p_2$ are the perpendiculars on the tangent from two fixed points. Then

$$m\frac{dp_1}{p_1} + n\frac{dp_2}{p_2} = 0;$$

but we have seen, p. 179, that $\frac{dp_1}{p_1 d\theta}$ is the cotangent of the angle made with the tangent by the radius vector from the first point; hence for the trajectory

$$m\frac{p_1}{dp_1} + n\frac{p_2}{dp_2} = 0; \quad p_1{}^n p_2{}^m = \text{C}.$$

295. We proceed next to give some account of the method of elliptic co-ordinates, which, although chiefly useful in space of three dimensions, may occasionally be applied with advantage to plane curves. In this system two fixed points being given, the position of any other is determined by the intersection of an ellipse and hyperbola having the fixed points for foci. The constant distance between the foci we shall denote by $2c$, and the primary semi-axes of any ellipse and hyperbola of the system by μ, ν. Then, to pass from rectangular co-ordinates, we have

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - c^2} = 1; \quad \frac{x^2}{\nu^2} - \frac{y^2}{c^2 - \nu^2} = 1;$$

$$cx = \mu\nu; \quad cy = \sqrt{\{(\mu^2 - c^2)(c^2 - \nu^2)\}}.$$

296. *To express the element of the arc of a curve by elliptic co-ordinates.*

Differentiating the equations just written, on the supposition that μ is constant, we get for the arc of the ellipse whose semi-axis is μ,

$$d\sigma = \sqrt{\left(\frac{\mu^2 - \nu^2}{c^2 - \nu^2}\right)}\, d\nu;$$

* The first Example of this Article is taken from a note by Mr. M. Roberts, Liouville, x. 251. For the second Example, as also for the following Articles on Elliptic Co-ordinates, I am indebted to the kindness of Mr. W. Roberts.

similarly for the arc of a hyperbola whose semi-axis is ν,

$$d\sigma' = \sqrt{\left(\frac{\mu^2 - \nu^2}{\mu^2 - c^2}\right)} d\mu;$$

and since these cut at right angles we have for the element of the arc of any other curve given by an equation between μ and ν (as we might also have obtained directly by differentiating, considering both μ and ν variable),

$$ds^2 = (\mu^2 - \nu^2)\left\{\frac{d\mu^2}{\mu^2 - c^2} + \frac{d\nu^2}{c^2 - \nu^2}\right\}.$$

The reader will find no difficulty in obtaining by this formula the arc of the Cartesian oval, $\mu = a\nu + b$, which will obviously be an ultra-elliptic function.

297. *To find the element of the area of a curve by elliptic co-ordinates.*

Considering as the element of the area the rectangle formed by the element of an ellipse, and that of a hyperbola of the system, we have the area

$$= \iint \frac{(\mu^2 - \nu^2)\, d\mu d\nu}{\sqrt{\{(\mu^2 - c^2)\,(c^2 - \nu^2)\}}}.$$

Thus we can readily find the space inclosed between two confocal ellipses, μ_1, μ_2, and two confocal hyperbolæ, ν_1, ν_2, viz.:

$$\int_{\mu_2}^{\mu_1} \frac{\mu^2 d\mu}{\sqrt{(\mu^2 - c^2)}} \cdot \int_{\nu_2}^{\nu_1} \frac{d\nu}{\sqrt{(c^2 - \nu^2)}} - \int_{\mu_2}^{\mu_1} \frac{d\mu}{\sqrt{(\mu^2 - c^2)}} \cdot \int_{\nu_2}^{\nu_1} \frac{\nu^2 d\nu}{\sqrt{(c^2 - \nu^2)}}.$$

298. *To find an orthogonal trajectory by elliptic co-ordinates.*

If the curve were given by a relation

$$\mathrm{P}d\sigma + \mathrm{Q}d\sigma' = 0,$$

$d\sigma$, $d\sigma'$ denoting, as before, the elements of the ellipse and hyperbola, it is plain that the orthogonal trajectory is given by the equation

$$\mathrm{P}d\sigma' - \mathrm{Q}d\sigma = 0.$$

Hence when the curve is given by the differential equation

$$\mathrm{M}d\mu + \mathrm{N}d\nu = 0,$$

the trajectory is given by the equation

$$\frac{d\mu}{\mathrm{M}\,(\mu^2 - c^2)} - \frac{d\nu}{\mathrm{N}\,(c^2 - \nu^2)} = 0.$$

Ex. *To find the orthogonal trajectory of the Cartesian ovals, $\mu + m\nu = a$, where a is given, and m is the variable parameter.*

We have

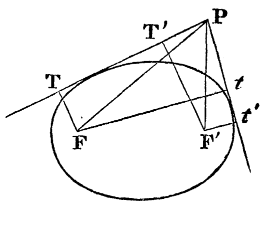
$$\frac{a-\mu}{\nu} = m; \quad \frac{d\mu}{a-\mu} + \frac{d\nu}{\nu} = 0;$$

hence for the trajectory,

$$\frac{(a-\mu)\,d\mu}{\mu^2 - c^2} - \frac{\nu d\nu}{c^2 - \nu^2} = 0,$$

the integration of which presents no difficulty.

299. *To find the relations which subsist between the elliptic co-ordinates of any point on a tangent to a given ellipse, $\mu = \mu_1$.*



It can be proved without much trouble that if we take on the tangent PT a portion PM = PF, and on Pt, PM′ = PF′, then MM′ = $2\mu'$, or the axis major of the ellipse. Hence if μ, ν be the elliptic co-ordinates of the point P, 2θ the angle between the tangents,

$$(\mu + \nu)^2 + (\mu - \nu)^2 - 2(\mu^2 - \nu^2)\cos 2\theta = 4\mu_1^2,$$
$$\mu^2 \sin^2\theta + \nu^2 \cos^2\theta = \mu_1^2;$$

an equation which expresses the relation between the co-ordinates of any point on the tangent and the angle which that tangent makes with the hyperbola through the point.

From this equation then

$$\tan\theta = \sqrt{\left(\frac{\mu_1^2 - \nu^2}{\mu^2 - \mu_1^2}\right)},$$

but also

$$\tan\theta = \frac{d\sigma}{d\sigma_1} = \sqrt{\left(\frac{\mu^2 - c^2}{c^2 - \nu^2}\right)}\,\frac{d\nu}{d\mu}.$$

Equating these values, we have for the differential equation of any tangent to the ellipse $\mu = \mu_1$,

$$\frac{d\mu}{\sqrt{\{(\mu^2 - c^2)\,(\mu^2 - \mu_1^2)\}}} \pm \frac{d\nu}{\sqrt{\{(c^2 - \nu^2)\,(\mu_1^2 - \nu^2)\}}} = 0.$$

Now this equation is of the well-known form which Euler proved to be algebraically integrable (Novi Com. Petrop. vi. and vii.);

and we see here how the same result appears from geometrical considerations, since we can obtain an algebraic relation between the μ and ν of any point on the tangent, by making in its x and y equation the substitutions of Art. 295.

300. *To find the equation of the involute of an ellipse.*

This is only to find the orthogonal trajectory of its tangents, whose differential equation is given in the last Article. Hence, by Art. 298, we have for the involute,

$$\sqrt{\left(\frac{\mu^2 - \mu_1^2}{\mu^2 - c^2}\right)} d\mu \mp \sqrt{\left(\frac{\mu_1^2 - \nu^2}{c^2 - \nu^2}\right)} d\nu = 0,$$

an equation which, the variables being separated, can be integrated by elliptic functions, and gives a transcendental relation between the μ and ν of any point of the involute.

301. In general, if dt be the element of the sum of arc and tangent to the ellipse, $\mu = \mu'$; then plainly $dt = \sin\theta d\sigma + \cos\theta d\sigma'$;

$$dt = \sqrt{\left(\frac{\mu_1^2 - \nu^2}{c^2 - \nu^2}\right)} d\nu \pm \sqrt{\left(\frac{\mu^2 - \mu_1^2}{\mu^2 - c^2}\right)} d\mu,$$

an equation which at once points to the possibility of finding an elliptic function of the second kind differing by an algebraic quantity from the sum of two given ones. It is obvious, too, that the involute must be given by the equation $dt = 0$.

302. We close this Chapter with some notice of a problem first solved by Euler, viz., *to find the curve whose n^{th} evolute shall be similar to itself*. For this purpose Euler defines a curve by the relation between the radius of curvature, R, and the angle θ made by that radius with a fixed line. From a relation of this kind, $R = f(\theta)$, we can obtain the x and y equation; for we have

$$ds = Rd\theta, \quad dx = ds\cos\theta, \quad dy = ds\sin\theta;$$

so that we can express x and y in terms of θ, and afterwards, by elimination, obtain the equation of the curve.

Now let us imagine that at the point where any radius of curvature meets the evolute, the radius of curvature of the evolute be drawn; and again, a radius of curvature to *its* evolute where that radius meets it, and so on: and let R', θ'; R'', θ'', &c. corres-

pond to the successive evolutes. Then, since each radius of curvature is perpendicular to the preceding one, we have

$$d\theta = d\theta' = d\theta'', \text{ \&c.},$$

and we have

$$dR = ds' = R'd\theta',$$

or

$$R' = \frac{dR}{d\theta}; \text{ and } R'' = \frac{dR'}{d\theta} = \frac{d^2R}{d\theta^2}, \text{ \&c.}$$

We can secure that the n^{th} evolute shall be similar to the original curve by taking $R_n = aR$; whence $\frac{d^nR}{d\theta^n} = aR$, a linear equation easily integrable. For example, when $n = 1$, the curve, whose evolute is similar to itself, is given by the equation $R = Ce^{a\theta}$.

Hence

$$x = \int Ce^{a\theta} \cos\theta d\theta = \frac{Ce^{a\theta}}{1 + a^2}(a\cos\theta + \sin\theta) + A,$$

$$y = \int Ce^{a\theta} \sin\theta d\theta = \frac{Ce^{a\theta}}{1 + a^2}(a\sin\theta - \cos\theta) + B.$$

From these equations

$$\tan\omega = \frac{y - B}{x - A} = \frac{a\sin\theta - \cos\theta}{a\cos\theta + \sin\theta},$$

or the angle made with the axis of x by the radius vector from the point $(x = A, y = B)$ differs from θ by the constant angle $\cot^{-1}a$. Squaring the equations, and adding, we have

$$\{(x - A)^2 + (y - B)\}^{\frac{1}{2}} = \frac{C}{\sqrt{(1 + a^2)}} e^{a\theta}.$$

It follows, then, that by changing the origin to the point $x = A$, $y = B$, and measuring ω from a line inclined to the axis of x at the angle just mentioned, the equation of the curve becomes

$$\rho = C'e^{a\omega},$$

the general equation of a logarithmic spiral.

303. This is Euler's solution of the problem, but it does not appear to be the most general solution, since it assumes that the two points whose radii of curvature are proportional lie on the same normal to the original curve. But more generally, if the

original curve be $R = f(\theta)$, it will suffice that the evolute should be $\frac{dR}{d\theta} = R' = af(\theta + \alpha)$; and the equation

$$f'(\theta) = af(\theta + \alpha)$$

is solved by taking $R = \Sigma C e^{m\theta}$, m being one of the roots of the equation $m = ae^{m\alpha}$. Thus the curve $R = Ce^{m\theta} \cos n\theta$ has an evolute similar to itself, an equation which gives the logarithmic spiral when n is taken $= 0$.

304. The problem *to find a curve whose evolute shall be similar to itself, but in an inverse position,* can be reduced to the former. For in this case the second evolute will be similar, and in the same position, as the original curve; but its radii of curvature increase as those of the original curve diminish; hence Euler finds the equation

$$\frac{d^2R}{d\theta^2} + a^2R = 0.$$

Or the same thing will appear from the method used in the last Article; for we must have

$$f'(\theta) = af(\alpha - \theta),$$
$$f''(\theta) = -af'(\alpha - \theta) = -a^2 f(\theta),$$

as may be seen by substituting $\alpha - \theta$ for θ in the given equation. The solution of this equation is

$$R = C\cos(a\theta + \beta),$$

or simply $= C\cos a\theta$, by merely altering the line from which θ is measured; whence

$$x = \frac{C}{2(a+1)}\sin(a+1)\theta + \frac{C}{2(a-1)}\sin(a-1)\theta,$$
$$y = \frac{C}{2(a-1)}\cos(a-1)\theta - \frac{C}{2(a+1)}\cos(a+1)\theta,$$

where the omission of the arbitrary constants is only equivalent to a suitable change in the position of the origin, and the equations are immediately recognised as reducible to these given (p. 211) for a hypocycloid. When a is less than 1, the two terms in y have the same sign, and the curve is an epicycloid. When $a = 1$ it can in like manner be proved that the curve is a

cycloid. The question to find the curve whose n^{th} evolute shall be similar to the original curve, but in an inverted position, is treated similarly.

305. We conclude this subject with the following theorem, due to John Bernouilli:

If there be any curve BC, *the tangents at the extremities of which are at right angles, and it be developed beginning at* B, *and if the involute* BD *be again developed, beginning at* D, *and so on* ad infinitum, *then the ultimate figure of the curve will be a cycloid.*

Since we have proved that for the evolute $R' = \frac{dR}{d\theta}$, it is plain that for the involute $R_1 = \int R d\theta$, and that the R of the next involute will be the integral of this again, and so on. From the conditions of the problem, the integrals are to be made to vanish alternately for $\theta = 0$ and $\theta = \frac{\pi}{2}$.

We may then assume for R_1 the first radius which vanishes for $\theta = \frac{\pi}{2}$, an expansion

$$A_1 \cos\theta + A_3 \cos 3\theta + A_5 \cos 5\theta \ldots + A_i \cos i\theta + \&c.,$$

determining A_i if $R_1 = f(\theta)$ by the equation

$$A_2 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(\theta) \cos i\theta d\theta.$$

It follows, then, that

$$R_2 = A_1 \sin\theta + \frac{A_3}{3} \sin 3\theta + \&c.$$

$$R_3 = A_1 \cos\theta + \frac{A_3}{3^2} \cos 3\theta + \&c.$$

all the terms, except the first, being affected with constantly increasing divisors, so that ultimately $R_\infty = A_1 \sin\theta$, or else $A_1 \cos\theta$, either of which is the equation of a cycloid (Art. 304).*

* The matter of the last four Articles has been taken from Lacroix's Integral Calculus, ii. 460, Gregory's Examples, p. 456; Arts. 303 and 305 I have adopted, with a little misgiving, from a memoir by M. Puiseux (Liouville, ix. 377). One would imagine from M. Puiseux's paper, that he was not acquainted with what Euler had done on this subject. On this proof see further the last note at the end.

NOTES.

Conics, Page 142.

I wish to add here demonstrations of Art. 154 (with which I was not acquainted at the time the former Part of this Treatise was published), taken from Mr. Boole's very interesting papers on Linear Transformations (Cambridge Mathematical Jour., iii. 1, 106, and New Series, vi. 87). Let us suppose that the quantity $Ax^2 + 2Bxy + Cy^2$ is transformed by linear substitutions into $A'\xi^2 + 2B'\xi\eta + C'\eta^2$; now we know that the quantity $x^2 + 2xy\cos\omega + y^2$ is at the same time transformed into $\xi^2 + 2\xi\eta\cos\omega' + \eta^2$, since either is the expression for the square of the distance of any point from the origin. It follows then that

$$Ax^2 + 2Bxy + Cy^2 + h(x^2 + 2xy\cos\omega + y^2) =$$
$$A'\xi^2 + 2B'\xi\eta + C'\eta^2 + h(\xi^2 + 2\xi\eta\cos\omega' + \eta^2).$$

But if h be so taken as to make the first side of the equation a perfect square, the second must be one also. The condition that the first side should be a perfect square is

$$(A + h)(C + h) = (B + h\cos\omega)^2;$$

or h must be one of the roots of the equation

$$h^2\sin^2\omega + (A + C - 2B\cos\omega)h + AC - B^2 = 0.$$

But comparing this with the similar equation derived from the second side of the equation, we must have

$$\frac{A + C - 2B\cos\omega}{\sin^2\omega} = \frac{A' + C' - 2B'\cos\omega'}{\sin^2\omega'},$$

$$\frac{AC - B^2}{\sin^2\omega} = \frac{A'C' - B'^2}{\sin^2\omega'};$$

equations agreeing with those obtained, *Conics*, Art. 154.

These equations suffice to determine the lengths of the axis of the conic,

$$Ax^2 + 2Bxy + Cy^2 = 1,$$

referred to axes inclined at any angle ω. For when the equation is transformed to the axes, we shall have $B'=0$, $\omega'=90^\circ$. Hence the equations just written, since they give the sum and the product of A', C', afford a quadratic for determining the length of the axes.

If it be desired to find the substitution necessary to transform the given equation to the axes, we have only to take the two equations

$$Ax^2 + 2Bxy + Cy^2 = A'\xi^2 + C'\eta^2,$$
$$x^2 + 2xy\cos\omega + y^2 = \xi^2 + \eta^2,$$

to multiply the second either by A' or C', to subtract and take the square root, when we have the expression for the new co-ordinates in terms of the old.

The whole of Mr. Boole's papers are well worthy the student's attention.

Page 58.

I thought it unnecessary to remark that the same investigations concerning poles and polars can be applied to tangential co-ordinates; and that thus every right line has a pole, a polar curve of the second, third, &c. class; and, finally, a polar curve of the $(n-1)^{st}$ class, touched by the n tangents at the points where the right line meets the curve. One fact, however, may cause this remark to seem deserving of a little more development. The centre of a conic section is, as we know, the pole of the line at infinity with respect to the curve. When we seek for the analogous point in curves of higher *degree*, we find that the line at infinity has several poles, all equally entitled to be considered the analogous point required: but when we seek the pole of a line with regard to a curve of higher *class*, by this tangential method we find but a single point. It may be interesting then to examine what properties such a pole of a line possesses, and more especially in the case when the line is at infinity. The method of Art. 61 applies, word for word, to tangential co-ordinates, only that now $\lambda:\mu$ denotes the ratio of the sines of the angles into which the angle between the two lines xyz, $x'y'z'$ is divided by the line $\lambda x+\mu x'$, &c. It is proved then, as in the text, that the point R,

$$x\left(\frac{dU}{dx}\right)_1 + y\left(\frac{dU}{dy}\right)_1 + z\left(\frac{dU}{dz}\right)_1 = 0,$$

possesses the property

$$\Sigma\left(\frac{\mu}{\lambda}\right) = \Sigma\left(\frac{\sin RPR_1}{\sin R_1PO}\right) = 0,$$

where P is a variable point on the given line; R_1, R_2, &c. the points of contact of tangents from the point P, O any fixed point on the given line. Thus for a curve of the second class the relation is

$$\frac{\sin RPR_1}{\sin R_1PO} + \frac{\sin RPR_2}{\sin R_2PO} = 0,$$

that is to say, "if from any point P, on a fixed line OP, we draw tangents, PR_1, PR_2, to a conic, and draw PR so that $\{P \cdot OR_1RR_2\}$ shall be a harmonic pencil, then OR passes through a fixed point." This is the fundamental definition of pole and polar with regard to a conic considered as a curve of the second class.

We may write the relation

$$\Sigma\left(\frac{\sin RPR_1}{\sin R_1PO}\right) = 0 \text{ in the form } \Sigma\left(\frac{M_1R_1}{R_1O_1}\right) = 0,$$

where M_1 is the foot of the perpendicular from R_1 on the line RP, and O_1 the foot of the perpendicular from the same point on the line OP. Now let the line OP go off to infinity, then all the denominators in this latter sum tend to equality, and we have simply $\Sigma(M_1R_1) = 0$; or the sum vanishes of the perpendiculars let fall from the points of contact of any system of parallel tangents on a parallel line through R. In other words then, *the centre of mean distances of the points of contact of any system of parallel tangents to a given curve is a fixed point, which may be regarded as the centre of the curve.* Thus in a conic the middle point of the line joining the points of contact of parallel tangents is a fixed point; in a curve of the third class the centre of gravity of the triangle formed by them, &c. This theorem is due to M. Chasles (Quetelet, vi. 8.), but I believe that the point of view from which it is here presented is the best from which to consider this and similar theorems.

On Elimination, Page 20.

In order to save the junior reader the trouble of referring to other works, I have put together here those elementary principles with regard to elimination, of which use has been made in the preceding pages; but it will be seen that I am very far from attempting to give a complete account of what is known on the subject. I am myself still but imperfectly acquainted with the improvements which this branch of analysis has of late years received; and at any rate the topic is too extensive to be satisfactorily treated in an Appendix to a work on a

different subject. We commence by a discussion of equations of the first degree between any number of variables.

I. *Linear Equations.*

(1.) The result of the elimination of n variables between n linear and homogeneous equations is called the *determinant* of that system of equations. Let us commence with the simplest case, when there are but two equations:

$$A_1x + B_1y = 0, \quad A_2x + B_2y = 0.$$

The elimination can be performed by multiplying the first equation by B_2, the second by B_1, and subtracting; or else by multiplying the first by A_2, and the second by A_1. In either case we find

$$A_1B_2 - A_2B_1 = 0,$$

which, for brevity, we shall write $(A_1B_2) = 0$, leaving the reader to supply the negative term: and in this notation it is plain that $(A_1B_2) = -(A_2B_1)$. We should have obtained the same result if we had eliminated in like manner between the equations

$$A_1x + A_2y = 0, \quad B_1x + B_2y = 0.$$

(2.) The process is precisely the same, if we were given the two equations

$$A_1x + A_2y + A_3z = 0, \quad B_1x + B_2y + B_3z = 0;$$

and if it were required to solve for x and y in terms of z. Proceeding precisely as before, we have

$$(A_1B_2)\,x + (A_3B_2)\,z = 0, \quad (A_1B_2)\,y + (A_1B_3)\,z = 0.$$

Substituting these values of x and y in the original equations, we get the system of equations identically true:

$$A_1(A_2B_3) + A_2(A_3B_1) + A_3(A_1B_2) = 0;$$
$$B_1(A_2B_3) + B_2(A_3B_1) + B_3(A_3B_2) = 0.$$

It is proved in like manner that

$$A_1(B_2C_1) + B_1(C_2A_1) + C_1(A_2B_1) = 0;$$
$$A_2(B_2C_1) + B_2(C_2A_1) + C_2(A_2B_1) = 0.$$

(3.) Next let it be required to eliminate between the equations

$$A_1x + B_1y + C_1z = 0, \quad A_2x + B_2y + C_2z = 0, \quad A_3x + B_3y + C_3z = 0.$$

Multiply the first by (A_2B_3), the second by (A_3B_1), the third by (A_1B_2), and add; then by the last paragraph the coefficients of x and y vanish, and we have

$$C_1(A_2B_3) + C_2(A_3B_1) + C_3(A_1B_2) = 0;$$

or writing at full length,

$$C_1A_2B_3 - C_1A_3B_2 + C_2A_3B_1 - C_2A_1B_3 + C_3A_1B_2 - C_3A_2B_1 = 0,$$

which is the determinant of the system of equations.

This determinant may also be written in either of the forms

$$A_1(B_2C_3) + A_2(B_3C_1) + A_3(B_1C_2), \text{ or } B_1(C_2A_3) + B_2(C_3A_1) + B_3(C_1A_2).$$

We shall use for this determinant the abbreviation $(A_1B_2C_3)$; but it is proper to mention that the ordinary method of writing the determinant, which is the result of elimination between the equations

$$\begin{aligned} A_1x + B_1y + C_1z + D_1w + \&c. &= 0,\\ A_2x + B_2y + C_2z + D_2w + \&c. &= 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ A_nx + B_ny + C_nz + D_nw + \&c. &= 0, \end{aligned}$$

is

$$\begin{vmatrix} A_1, & B_1, & C_1, & D_1, \ldots \\ A_2, & B_2, & C_2, & D_2, \ldots \\ \cdots & \cdots & \cdots & \cdots \\ A_n, & B_n, & C_n, & D_n, \ldots \end{vmatrix}$$

as we have written at p. 188.

(4.) We might in like manner eliminate between the three equations

$$A_1x + A_2y + A_3z = 0, \quad B_1x + B_2y + B_3z = 0, \quad C_1x + C_2y + C_3z = 0;$$

and we should find by the same process as before

$$A_1(B_2C_3) + B_1(C_2A_3) + C_1(A_2B_3) = 0;$$

but this is precisely the same quantity as the determinant found in the last paragraph. Hence

$$\begin{vmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{vmatrix} = \begin{vmatrix} A_1, & B_1, & C_1. \\ A_2, & B_2, & C_2. \\ A_3, & B_3, & C_3. \end{vmatrix}$$

(5.) We proceed in precisely the same manner to solve the equations

$$\begin{aligned} A_1x + A_2y + A_3z + A_4w &= 0,\\ B_1x + B_2y + B_3z + B_4w &= 0,\\ C_1x + C_2y + C_3z + C_4w &= 0. \end{aligned}$$

Multiplying the first by (B_2C_3), the second by (C_2A_3), the third by (A_2B_3), and adding, we have

$$(A_1B_2C_3)\,x + (A_4B_2C_3)\,w = 0.$$

Similarly

$$(A_1B_2C_3)\,y + (B_4C_3A_1)\,w = 0, \quad (A_1B_2C_3)\,z + (C_4A_1B_2)\,w = 0.$$

Substituting these values in the given equations, and observing that $(A_4B_2C_3) = -(A_2B_3C_4)$, &c., we have the identical relations

$$\begin{aligned}
&A_1(A_2B_3C_4) - A_2(A_3B_4C_1) + A_3(A_4B_1C_2) - A_4(A_1B_2C_3) = 0,\\
&B_1(A_2B_3C_4) - B_2(A_3B_4C_1) + B_3(A_4B_1C_2) - B_4(A_1B_2C_3) = 0,\\
&C_1(A_2B_3C_4) - C_2(A_3B_4C_1) + C_3(A_4B_1C_2) - C_4(A_1B_2C_3) = 0.
\end{aligned}$$

(6.) Suppose next that it were required to eliminate between the equations

$$\begin{aligned}
A_1x + B_1y + C_1z + D_1w &= 0,\\
A_2x + B_2y + C_2z + D_2w &= 0,\\
A_3x + B_3y + C_3z + D_3w &= 0,\\
A_4x + B_4y + C_4z + D_4w &= 0.
\end{aligned}$$

Multiplying the first equation by $(A_2B_3C_4)$, the second by $-(A_3B_4C_1)$, the third by $(A_4B_1C_2)$, the fourth by $-(A_1B_2C_3)$, and adding; then the coefficients of x, y, z vanish identically, and the required determinant is

$$D_1(A_2B_3C_4) - D_2(A_3B_4C_1) + D_3(A_4B_1C_2) - D_4(A_1B_2C_3) = 0;$$

or, writing at full length,

$$\begin{aligned}
&A_1B_2C_3D_4 - A_1B_3C_2D_4 + A_2B_3C_1D_4 - A_2B_1C_3D_4 + A_3B_1C_2D_4 - A_3B_2C_1D_4\\
&+ A_1B_4C_2D_3 - A_1B_2C_4D_3 + A_4B_2C_1D_3 - A_4B_1C_2D_3 + A_2B_1C_4D_3 - A_2B_4C_1D_3\\
&+ A_3B_4C_1D_2 - A_3B_1C_4D_2 + A_4B_1C_3D_2 - A_4B_3C_1D_2 + A_1B_3C_4D_2 - A_1B_4C_3D_2\\
&+ A_2B_4C_3D_1 - A_2B_3C_4D_1 + A_4B_3C_2D_1 - A_4B_2C_3D_1 + A_3B_2C_4D_1 - A_3B_4C_2D_1\\
&= 0.
\end{aligned}$$

(7.) The reader can now find no difficulty in extending this process to any greater number of equations; and he will see that the general expression for a determinant is $\Sigma \pm A_1B_2C_3D_4$, &c., where each product must include all the varieties of the n letters, and of the n suffixes, without repetition or omission, and where the sign + or − is to be given to the product according as it is obtained from the first term by an even or an odd number of permutations of suffixes. Thus in the last example, $A_1B_3C_2D_4$ differs from $A_1B_2C_3D_4$ by a permutation of the suffixes of BC, and therefore has an opposite sign; the third term differs from the second by a permutation of the suffixes of AC, and therefore has an opposite sign; but it has the same sign as the first term, since it can only be obtained from it by twice interchanging suffixes. It may be deduced from this rule that *a cyclic interchange of suffixes alters the sign when the number of terms in the product is even, but not so when the number of terms in the product is odd.* Thus $A_1B_2C_3$, $A_2B_3C_1$, in (3) have the same sign, since the one can only be derived from the other by a double permutation of suffixes; but $A_1B_2C_3D_4$, $A_2B_3C_1D_4$ have opposite signs, since the one is derived from the other by a triple permutation of suffixes. The reader sees also how each determinant may be formed from

that of the order preceding; that the next determinant, for example, will be

$$A_1(B_2C_3D_4E_5) + A_2(B_3C_4D_5E_1) + A_3(B_4C_5D_1E_2) + A_4(B_5C_1D_2E_3) + A_5(B_1C_2D_3E_4);$$

but the signs would be alternately + and −, if the determinant were of an even order.

(8.) We have seen that the determinant

$$\begin{vmatrix} A_1, & B_1, & C_1, & D_1, \\ A_2, & B_2, & C_2, & D_2, \\ A_3, & B_3, & C_3, & D_3, \\ A_4, & B_4, & C_4, & D_4, \end{vmatrix}$$

is formed by adding, with the proper signs, all the possible products which can be obtained by taking one constituent from each horizontal and each vertical row. It follows then, from the symmetry of this proceeding, that what we have already proved for determinants of the second and third orders is true in general, or that *the value of a determinant is not altered by writing the horizontal rows vertically, and the vertical horizontally;* that, for example, the same result is found whether we eliminate between the four equations in (6), or between the four,

$$\begin{aligned} A_1x + A_2y + A_3z + A_4w &= 0, \\ B_1x + B_2y + B_3z + B_4w &= 0, \\ C_1x + C_2y + C_3z + C_4w &= 0, \\ D_1x + D_2y + D_3z + D_4w &= 0. \end{aligned}$$

(9.) *If in a determinant any two vertical or two horizontal rows be the same, the determinant vanishes identically.*

Suppose, for example, that in the system just written, $C_1 = D_1$, $C_2 = D_2$, &c., then it is evident that the last two equations will be the same, and that the values of x, y, z, obtained from the first three equations, will, when substituted in the last, make it vanish.

Since every term of a determinant contains one, and but one, constituent from each horizontal or vertical row, it follows that *if the whole of a vertical or horizontal row be multiplied by any quantity, the determinant is multiplied by that quantity.*

(10.) For the same reason, *if each term in any vertical or horizontal row be resolved into the sum of two others, the determinant can be resolved into the sum of two others.*

Thus

$$\begin{vmatrix} A_1 + a_1, & B_1, & C_1 \\ A_2 + a_2, & B_2, & C_2 \\ A_3 + a_3, & B_3, & C_3 \end{vmatrix} = \begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix} + \begin{vmatrix} a_1, & B_1, & C_1, \\ a_2, & B_2, & C_2, \\ a_3, & B_3, & C_3. \end{vmatrix}$$

Since the left-hand side of the equation is

$$(A_1 + a_1)(B_2C_3) + (A_2 + a_2)(B_3C_1) + (A_3 + a_3)(B_1C_2).$$

The reader will have no difficulty in seeing, in general, how to resolve into several others a determinant, each of whose constituents is the sum of several terms.

(11.) We conclude with one of the most important theorems concerning determinants, that *the product of two determinants is a determinant;* that the product of the determinant $(A_1B_2C_3$ &c.) by the determinant $(a_1b_2c_3$ &c.) is the determinant, whose first vertical row is $A_1a_1 + B_1b_1 +$ &c., $A_1a_2 + B_1b_2 +$ &c., $A_1a_3 +$ &c., whose second is $A_2a_1 + B_2b_1 +$ &c., $A_2a_2 + B_2b_2 +$ &c., &c.

The proof which we shall give for determinants of the third order will apply in general. Thus

$$\begin{vmatrix} A_1a_1 + B_1b_1 + C_1c_1, & A_2a_1 + B_2b_1 + C_2c_1, & A_3a_1 + B_3b_1 + C_3c_1, \\ A_1a_2 + B_1b_2 + C_1c_2, & A_2a_2 + B_2b_2 + C_2c_2, & A_3a_2 + B_3b_2 + C_3c_2, \\ A_1a_3 + B_1b_3 + C_1c_3, & A_2a_3 + B_2b_3 + C_2c_3, & A_3a_3 + B_3b_3 + C_3c_3, \end{vmatrix}$$

is the result of eliminating xyz between the equations

$$\begin{aligned} a_1S_1 + b_1S_2 + c_1S_3 &= 0, \\ a_2S_1 + b_2S_2 + c_2S_3 &= 0, \\ a_3S_1 + b_3S_2 + c_3S_3 &= 0, \end{aligned}$$

where, for brevity, we write

$$\begin{aligned} S_1 &= A_1x + A_2y + A_3z, \\ S_2 &= B_1x + B_2y + B_3z, \\ S_3 &= C_1x + C_2y + C_3z. \end{aligned}$$

Now this elimination may be performed at once by eliminating S_1, S_2, S_3; hence we see that $(a_1b_2c_3)$ must be a factor in the result. But also a system of values of xyz can be found to satisfy simultaneously the given equations if a system can be found to satisfy simultaneously the equations $S_1 = 0$, $S_2 = 0$, $S_3 = 0$. Therefore $(A_1B_2C_3) = 0$, which is the condition that the latter should be possible, must also be a factor in the result. And we can see that the degree of the result in the coefficients is exactly the same as that of the product of these two quantities; the result is therefore $(A_1B_2C_3)(a_1b_2c_3)$.

This theorem, for the multiplication of determinants, may with ad-

vantage be expressed as follows:—If in any system of equations of the first degree,

$$a_1x + b_1y + c_1z + \&c. = 0, \quad a_2x + b_2y + \&c. = 0, \&c.,$$

the variables be transformed by any linear substitution,

$$x = A_1\xi + B_1\eta + \&c., \quad y = A_2\xi + B_2\eta + \&c. \ \&c.,$$

then the determinant of the transformed system will be equal to the determinant of the original system ($a_1b_2c_3$ &c.), multiplied by what may be called the determinant of transformation ($A_1B_2C_3$ &c.)

(12.) The following example will suffice to make the reader understand the geometrical importance of the preceding theorem. The condition that the general equation of the second degree,

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0,$$

should represent two right lines, is the determinant

$$\begin{vmatrix} A, & B, & D, \\ B, & C, & E, \\ D, & E, & F. \end{vmatrix}$$

Now it is plain that if this condition be satisfied, the equation must still represent two right lines however the co-ordinates be transformed. But if we make any linear transformation,

$$\begin{aligned} x &= a_1\xi + b_1\eta + c_1\zeta, \\ y &= a_2\xi + b_2\eta + c_2\zeta, \\ z &= a_3\xi + b_3\eta + c_3\zeta, \end{aligned}$$

the condition that the new equation should represent two right lines is the determinant,

$$\begin{vmatrix} Aa_1 + Ba_2 + Da_3, & Ab_1 + Bb_2 + Db_3, & Ac_1 + Bc_2 + Dc_3, \\ Ba_1 + Ca_2 + Ea_3, & Bb_1 + Cb_2 + Eb_3, & Bc_1 + Cc_2 + Ec_3, \\ Da_1 + Ea_2 + Fa_3, & Db_1 + Eb_2 + Fb_3, & Dc_1 + Ec_2 + Fc_3, \end{vmatrix}$$

which, by the theorem just proved, is the product of ($a_1b_2c_3$) into the first-mentioned determinant, and therefore must vanish with this determinant, as is obvious from geometrical considerations.

It is less necessary to discuss determinants at greater length, because Mr. Spottiswoode* has lately made the subject accessible to the English reader. Those desirous of further information may consult the writers referred to by him.

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* Elementary Theorems relating to Determinants; published by Longman and Co., 1851.

(13.) I cannot, however, forbear adding another geometrical illustration, taken from a memoir by M. Joachimstal (Crelle, xl. 30). It is the analytical proof of the theorem proved geometrically (*Conics*, p. 318),

Let us write

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=\mathrm{A},\quad \frac{x''^2}{a^2}+\frac{y''^2}{b^2}-1=\mathrm{C},\quad \frac{x'''^2}{a^2}+\frac{y'''^2}{b^2}-1=\mathrm{F},$$

$$\frac{x''x'''}{a^2}+\frac{y''y'''}{b^2}-1=\mathrm{E},\quad \frac{x'x'''}{a^2}+\frac{y'y'''}{b^2}-1=\mathrm{D},\quad \frac{x'x''}{a^2}+\frac{y'y''}{b^2}-1=\mathrm{B}.$$

Let the area of the triangle whose vertices are $x'y'$, $x''y''$, $x'''y'''$, be M, then the determinant

$$\begin{vmatrix} \frac{x'}{a}, & \frac{y'}{b}, & 1 \\ \frac{x''}{a}, & \frac{y''}{b}, & 1 \\ \frac{x'''}{a}, & \frac{y'''}{b}, & 1 \end{vmatrix} = \pm\frac{2\mathrm{M}}{ab};\qquad \begin{vmatrix} \frac{x'}{a}, & \frac{y'}{b}, & -1 \\ \frac{x''}{a}, & \frac{y''}{b}, & -1 \\ \frac{x'''}{a}, & \frac{y'''}{b}, & -1 \end{vmatrix} = \mp\frac{2\mathrm{M}}{ab}.$$

But if we form by the preceding rules the product of these two determinants, we shall have

$$-\frac{4\mathrm{M}^2}{a^2b^2}=\begin{vmatrix} \mathrm{A}, & \mathrm{B}, & \mathrm{D}, \\ \mathrm{B}, & \mathrm{C}, & \mathrm{E}, \\ \mathrm{D}, & \mathrm{E}, & \mathrm{F}, \end{vmatrix}$$

whence

$$\mathrm{M}=\frac{ab}{2}(\mathrm{AE}^2+\mathrm{CD}^2+\mathrm{FB}^2-\mathrm{ACF}-2\mathrm{BDE})^{\frac{1}{2}}.$$

If now the triangle be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

$$\mathrm{A}=0,\quad \mathrm{C}=0,\quad \mathrm{F}=0;$$

and if g, h, k be the sides of the triangle; G, H, K, the parallel semi-diameters, it will be readily seen that

$$-2\mathrm{E}=\frac{(x''-x''')^2}{a^2}+\frac{(y''-y''')^2}{b^2}=\frac{g^2}{\mathrm{G}^2},\ \&\text{c}.$$

Consequently

$$\mathrm{M}=\tfrac{1}{4}ab\,\frac{ghk}{\mathrm{GHK}};$$

from which expression the value given (*Conics*, p. 318) for the radius of the circumscribing circle is at once derived.

II. *Elimination between two Equations.*

(1.) We proceed next to the case where the given equations are of any degree, but shall first, for simplicity, discuss at greater length the case where there are but two equations. We shall call the result of elimination between them the *resultant* of the equations. We have already given the resultant when both equations are of the first degree. We go on now to eliminate between two quadratics:

$$Ax^2 + Bxy + Cy^2 = 0, \quad A'x^2 + B'xy + C'y^2 = 0.$$

Multiply the first by A′, the second by A, and subtract, and we have

$$(BA')\,x + (CA')\,y = 0.$$

Multiply the first by C′, the second by C, and subtract, and we have

$$(CA')\,x + (CB')\,y = 0;$$

and eliminating between these linear equations, the final result is

$$(CA')^2 + (BA')\,(BC') = 0.$$

(2.) Elimination between two cubics,

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 = 0, \quad A'x^3 + B'x^2y + C'xy^2 + D'y^3 = 0,$$

is reduced in like manner to elimination between the two quadratics,

$$(BA')\,x^2 + (CA')\,xy + (DA')\,y^2 = 0, \quad (DA')\,x^2 + (DB')\,xy + (DC')\,y^2 = 0;$$

and by the previous paragraph the result is

$$\{(DA')^2 - (BA')\,(DC')\}^2 + \{(DA')\,(CA') - (BA')\,(DB')\}$$
$$\{(CA')\,(DC') - (DA')\,(DB')\} = 0.$$

Now it must be observed that

$$(BA')\,(DC') - (DB')\,(CA') = (DA')\,(BC').$$

Hence the result just given is divisible by (DA′); expanding therefore, and performing this division, we have finally

$$(DA')^3 - 2(DA')\,(BA')\,(DC') - (DA')\,(CA')\,(DB')$$
$$+ (CA')^2\,(DC') + (DB')^2\,(BA') + (BA')\,(BC')\,(DC') = 0.$$

(3.) In like manner, elimination between the two biquadratics,

$$Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 = 0,$$
$$A'x^4 + B'x^3y + C'x^2y^2 + D'xy^3 + E'y^4 = 0,$$

is reduced to elimination between the two cubics,

$$(BA')\,x^3 + (CA')\,x^2y + (DA')\,xy^2 + (EA')\,y^3 = 0,$$
$$(EA')\,x^3 + (EB')\,x^2y + (EC')\,xy^2 + (ED')\,y^3 = 0.$$

We apply the formula of the last paragraph; we find a result which is divisible by $(EA')^2$, and which gives finally

$$(EA')^4+3(BA')(DE')(EA')^2+2(CA')(CE')(EA')^2+(BE')(DA')(EA')^2$$
$$+(EA')\{(EB')^2(CA')+(DA')^2(EC')+3(CA')(DA')(ED')\}$$
$$+(EA')\{3(BA')(EB')(EC')+(CA')(CE')(DB')+(BA')(DE')(DB')\}$$
$$+2(BA')^2(ED')^2-2(BE')(DE')(CA')^2-2(BA')(DA')(CE')^2$$
$$+(BA')(BE')^3+(DE')(DA')^3+(CA')^2(CE')^2-(CA')(DA')(DB')(DE')$$
$$-(CE')(BA')(BD')(BE')+(BA')(BC')(DE')(DC')$$
$$+(CA')^2(DC')(DE')+(CE')^2(BC')(BA')-2(BA')(BC')(BE')(DE')$$
$$-2(DE')(DC')(DA')(BA')+(BA')(DE')(DB')^2=0.$$

This result, of course, by making $A'=0$, includes the result of eliminating between a biquadratic and a cubic; and so for other cases. The same process exactly is applicable to elimination between two equations of the fifth degree. The work, no doubt, would be laborious, and the result long, but this inconvenience could scarcely be avoided, however the elimination be performed.*

(4.) Although the process of elimination employed in the last three paragraphs is that which I have found most convenient in practice, it is open to the serious theoretical objection that it introduces into the result the irrelevant factors (DA'), $(EA')^2$, &c. The following mode of elimination, introduced, I believe, by Mr. Sylvester, is free from this objection. The process is to reduce all elimination to linear elimination. Thus in the case of the two quadratics of (1), if we multiply both equations by x and by y, we obtain the system

$$Ax^3+Bx^2y+Cy^2x=0,\qquad Ax^2y+Bxy^2+Cy^3=0,$$
$$A'x^3+B'x^2y+C'y^2x=0,\qquad A'x^2y+B'xy^2+C'y^3=0,$$

from which we can linearly eliminate x^3, x^2y, xy^2, y^3, as if they were independent unknowns, so that the result is the determinant

$$\begin{vmatrix} A, & B, & C, & 0, \\ A', & B', & C', & 0, \\ 0, & A, & B, & C, \\ 0, & A', & B', & C'. \end{vmatrix}$$

The same process applies whatever be the degree of the equations.

(5.) Elimination by symmetric functions will also give the result

* Since this note was written I have discovered that these eliminations have been given some time ago by Mr. Moon, Cambridge Mathematical Journal, iii. 183. Three or four of the terms in Mr. Moon's final result differ from mine. I have, in consequence, gone over the calculations again, without being able to detect any error in my work.

free from irrelevant factors, and will, moreover, exhibit clearly the degree of the result. Let the two equations be

$$\phi = \mathrm{A}x^m + \mathrm{B}x^{m-1}y + \mathrm{C}x^{m-2}y^2 + \&c. = 0,$$
$$\psi = \mathrm{A}'x^n + \mathrm{B}'x^{n-1}y + \mathrm{C}'x^{n-2}y^2 + \&c. = 0,$$

and let the roots of the first equation, solved for $x:y$, be α, β, γ, &c., and of the second α', β', γ', &c.; then if the equations have a common root, some one of the quantities, α, β, γ, &c., substituted in the second equation, must make it to vanish; and therefore the continued product of the results of these m substitutions must in this case be sure to vanish. The condition required is therefore this product, $(\psi\alpha)(\psi\beta)(\psi\gamma)$ &c. $= 0$. It appears then immediately that the result is of the m^{th} degree in the coefficients of the second equation. But the condition may be also written $(\phi\alpha')(\phi\beta')(\phi\gamma')$ &c. $= 0$, and is therefore of the n^{th} degree in the coefficients of the first equation. That the two forms are equivalent appears from considering that

$$\psi\alpha = (\alpha - \alpha')(\alpha - \beta') \; \&c., \quad \phi\alpha' = (\alpha' - \alpha)(\alpha' - \beta)(\alpha' - \gamma) \; \&c.,$$

and we have obviously

$$(\alpha - \alpha')(\alpha - \beta') \; \&c. \; (\beta - \alpha')(\beta - \beta') \; (\&c.)$$
$$= \pm (\alpha' - \alpha)(\alpha' - \beta) \; \&c. \; (\beta' - \alpha)(\beta' - \beta) \; \&c.$$

If we suppose each of the roots α, β, α', β', &c. to be multiplied by a factor, z, then since each of the mn factors $(\alpha - \alpha')$ is multiplied by z, it is evident that the resultant will be multiplied by z^{mn}. Or, in other words, if B, B′ contain a factor z, C, C′ contain a factor z^2, and so on, the entire result will be divisible by z^{mn}. Hence if B, B′ contain z in a degree not higher than the first, C, C′ in a degree not higher than the second, &c., then the resultant will contain z in the mn^{th} and lower degrees.

(6.) We have seen that if we suppose the equations resolved into the linear factors, $x - \alpha y = 0$, $x - \beta y = 0$, $x - \alpha' y = 0$, &c., then the final result is the product of the mn determinants, found by eliminating between any equation of the first and any of the second system. If now we suppose the variables transformed by any linear substitution, $x = \lambda\xi + \mu\eta$, $y = \lambda'\xi + \mu'\eta$, the effect is to multiply each of these determinants by the determinant of transformation $(\lambda\mu' - \lambda'\mu)$. We see, then, that if two equations be transformed by a linear substitution, the resultant of the transformed equations will be equal to that of the original multiplied by $(\lambda\mu' - \lambda'\mu)^{mn}$. The same thing might be seen by supposing α, α', the roots of one system, to be connected with a, a', the

roots of the other, by relations $\alpha = \dfrac{\lambda a + \mu}{\lambda' a + \mu'}$, $\alpha' = \dfrac{\lambda a' + \mu}{\lambda' a' + \mu'}$, when it can at once be verified that $\alpha - \alpha'$ is proportional to $(\lambda\mu' - \lambda'\mu)\ (a - a')$.

(7.) The most important example of the application of the preceding principles is the problem of finding the condition that a given equation

$$T = Ax^n + nBx^{n-1}y + \frac{n(n-1)}{1.2}Cx^{n-2}y^2 + \&c. = 0,$$

should have equal roots. This condition is often called the determinant of the equation. The coefficients are introduced on account of the symmetry which they give to the equations, and also because, since if $A = B = C$, &c., the equation just written will evidently have all its roots equal, the required condition must vanish on the same supposition, and we shall thus have a test of the accuracy of our results. The condition is obviously the result of elimination between $\dfrac{dT}{dx} = 0$, $\dfrac{dT}{dy} = 0$; it is therefore by (5) of the degree $2(n-1)$ in the coefficients, since each of these equations is of the $(n-1)^{st}$ degree; and if B contain a variable, z, in the first degree, C in the second, and so on, the result will be homogeneous in z, and of the $n(n-1)^{st}$ degree.

For example, to find the determinant of $Ax^2 + 2Bxy + Cy^2 = 0$, depends on elimination between the linear equations

$$Ax + By = 0, \quad Bx + Cy = 0,$$

and is therefore $B^2 - AC = 0$.

To find the determinant of $Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = 0$ depends on elimination between

$$Ax^2 + 2Bxy + Cy^2 = 0, \quad Bx^2 + 2Cxy + Dy^2 = 0,$$

and is therefore, by (1),

$$(AD - BC)^2 - 4(B^2 - AC)(C^2 - BD) = 0.$$

The determinant of $Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4 = 0$ is in like manner found to be

$$\begin{aligned} A^3E^3 &+ 36B^2C^2D^2 + 54AB^2CE^2 + 54A^2CD^2E + 108B^3CED + 108ABCD^3 \\ &+ 81AEC^4 = 64B^3D^3 + 54AC^3D^2 + 54B^2EC^3 + 27B^4E^2 + 27A^2D^4 \\ &+ 6AB^2D^2E + 12A^2BDE^2 + 18A^2E^2C^2 + 180ABC^2ED. \end{aligned}$$

(8.) In this shape the equation is too cumbrous to be of any practical utility. We owe to Messrs. Boole and Cayley the reduction of the

preceding to the form, of which use has been so frequently made in the preceding pages, viz.,

$$(AE - 4BD + 3C^2)^3 = 27(ACE + 2BCD - AD^2 - EB^2 - C^3)^2.$$

It is evident from (6) that the determinant of an equation is unchanged (or only changed by the introduction of a factor) by linear transformations of the variables; we know also that this determinant is the continued product of the squares of the differences of the roots of an equation. Now Mr. Cayley first showed that beside the determinant there are other derivatives from an equation (which he calls hyperdeterminants), which possess this property of being unaffected by linear transformation. Without entering into other modes of obtaining such functions (for which see the recent Papers of Messrs. Cayley, Boole, and Sylvester), it is plain from (6) that any symmetrical function of the roots of the equation, which is also a function of the differences of the roots, and into each term of which all the roots enter the same number of times, will possess this property. The latter condition is necessary, in order that, after transformation, all the terms should have the same denominator. Thus for a biquadratic, the function

$$(\alpha - \beta)^2 (\gamma - \delta)^2 + (\alpha - \gamma)^2 (\beta - \delta)^2 + (\alpha - \delta)^2 (\beta - \gamma)^2,$$

into each term of which each root enters twice, answers this description. And when expressed in terms of the coefficients, it gives Mr. Cayley's hyperdeterminant, $AE - 4BD + 3C^2$. It appears from the expression in terms of the roots, that this function will vanish when the equation has three equal roots. The hyperdeterminant

$$(ACE + 2BCD - AD^2 - EB^2 - C^3)$$

was discovered by Mr. Boole. It is expressed in terms of the roots as the sum, with proper signs, of six terms $(\alpha - \beta)^2 (\gamma - \delta)^2 (\alpha - \gamma) (\beta - \delta)$, thus answering to our description, and each root entering thrice into each term. Or it may be written more conveniently

$$\{(\alpha - \beta)(\gamma - \delta) - (\alpha - \gamma)(\delta - \beta)\} \{(\alpha - \gamma)(\delta - \beta) - (\alpha - \delta)(\beta - \gamma)\} \{(\alpha - \delta)(\beta - \gamma) - (\alpha - \beta)(\gamma - \delta)\},$$

a form both convenient for calculation of this function in terms of the coefficients, and also exhibiting at once the last theorem of Art. 205. This function vanishes also when the equation has three equal roots, and the two together afford the simplest conditions that such should be the case. The expression of the determinant, in terms of these two functions of the second and third degree in the coefficients, is due to Mr. Boole.

(9.) For the conditions that the given biquadratic should have two distinct pairs of equal roots, I refer to a paper (Cambridge and Dublin Math. Journal, iii. 171). The conditions there given were obtained by a kind of tentative process; by first forming the general expression of an equation complying with these conditions $(x-\alpha y)^2(x-\beta y)^2=0$, by expressing the coefficients in terms of α, β, and then eliminating $\alpha\beta$. It would appear, however, that the method of symmetric functions might have been used perhaps with more advantage for the same purpose. Thus if we form any symmetric function which shall vanish when $\alpha=\beta$ and $\gamma=\delta$, we shall have one of the conditions required. For example,

$$(\alpha+\beta-\gamma-\delta)\ \{(\alpha-\beta)^2-(\gamma-\delta)^2\}=0$$

is a symmetric function of the roots, and is the function

$$2B^3+A^2D-3ABC$$

given in the Paper cited. Similarly for the rest. I have only to add, that, in general, when an equation has two pairs of equal roots, we can readily form an equation which shall be satisfied by either of them, in addition to the two $T=0$, and $\frac{dT}{dx}=0$. For if we substitute in the equation $x+h$ for x, it becomes

$$T+\frac{dT}{dx}\frac{h}{1}+\frac{d^2T}{dx^2}\frac{h^2}{1\,.\,2}+\&c.=0.$$

Now if we give x the value of one of the equal roots, the first two terms vanish, and because there is a second pair of equal roots the remaining equation in h must have two equal roots. Hence if we form the condition that

$$\frac{d^2T}{dx^2}+\frac{d^3T}{dx^3}\frac{h}{3}+\frac{d^4T}{dx^4}\frac{h^2}{3\,.\,4}+\&c.=0,$$

considered as an equation in h, should have equal roots, this condition will be satisfied by any of the equal roots of the original equation. Thus if a biquadratic have two pairs of equal roots, either of them must satisfy

$$T=0,\quad \frac{dT}{dx}=0,\quad \left(\frac{d^3T}{dx^3}\right)^2=3\left(\frac{d^2T}{dx^2}\right)\left(\frac{d^4T}{dx^4}\right).$$

The last, being a quadratic equation, gives the equal roots at once. We see thus that if

$$Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4=0,$$

have two pairs of equal roots, these roots are given by the equation

$$A^2x^2+2ABxy+(3AC-2B^2)\,y^2=0;$$

but they are in like manner also given by the equation

$$(3CE - 2D^2)\,x^2 + 2DExy + E^2y^2 = 0.$$

These equations being identical, we have at once relations which must be satisfied by the coefficients, in order that the equation should have two pairs of equal roots. I have no doubt that this method of obtaining these relations, which only occurs to me as I write, might be generalized by a little attention. The reader will see the connexion between this and the theory of double tangents.

(10.) I refer to the Cambridge and Dublin Mathematical Journal, v. 152, &c. for the determinant of the equation of the fifth degree, and for some general considerations as to the formation of these functions. I only wish to add some observations as to the form of the determinant of an equation of the n^{th} degree, of which use has been made in the preceding pages. If we have found the condition that

$$T = Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \&c. = 0,$$

should have equal roots, and call this ϕ_n, it is obvious that on making $A = 0$ in the result we shall have the condition ϕ_{n-1} that

$$Bx^{n-1} + Cx^{n-2}y + Dx^{n-3}y^2 + \&c. = 0$$

should have equal roots. ϕ_n is of the degree $2(n-1)$ in the coefficients, while ϕ_{n-1} is only of the degree $2(n-2)$; hence there is still a factor of the second degree to be accounted for. But in general we may, as at p. 94, consider $\phi_n = 0$ as the equation of a curve to which T is a tangent, meeting it where it meets $\dfrac{dT}{dy}$. Hence, in particular, A is a tangent at the point AB; ϕ_n must therefore be of the form $A\psi + B^2\chi$. We see then, from what precedes, that the form of ϕ_n is

$$\phi_n = A\psi + B^2\phi_{n-1}.$$

Thus if it were required to find the determinant of

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 = 0,$$

we might, previous to calculation, write down the terms

$$B^2C^2 - 4B^3D - 4C^3A + AD\psi = 0;$$

since the result of making $A = 0$ must be $B^2(C^2 - 4BD)$, and of making $D = 0$ must, in like manner, be $C^2(B^2 - 4AC)$.

III. *Elimination between any number of Equations.*

A brief notice must suffice of the problem of elimination in general. Were we given three equations,

$$A x^m + B x^{m-1} y + C x^{m-1} z + \&c. = 0,$$
$$A' x^n + B' x^{n-1} y + C' x^{n-1} z + \&c. = 0,$$
$$A'' x^p + B'' x^{p-1} y + C'' x^{p-1} z + \&c. = 0,$$

it appears from (5) of the last section that if we solve for x and y in terms of z from the first pair of equations, we shall have mn sets of values of x and y. Some one of these substituted in the third equation will satisfy it; therefore the required condition is found by substituting each of the mn sets successively, and multiplying all the results together. We see thus that the coefficients of the third equation must enter in the degree mn into the result; and so in like manner those of the second will enter in the degree mp, and those of the first in the degree np. And, in general, if we have to eliminate between any number of equations, it is proved in like manner that the degree in which the coefficients of any one enter into the result is found by multiplying together the exponents of the degrees of all the rest.

The symmetric functions of the roots of the first two equations, which this method requires us to form, are obtained by Poisson's method. Assume a new variable, $t = \lambda x + \mu y$; eliminating x and y from this and the first two equations, we obtain an equation of the mn^{th} degree in t; form the sum of the k^{th} powers of the roots of this equation; then it is obvious that the coefficient of λ^k in this sum will be Σx^k, the coefficient of $\lambda^{k-1}\mu$ will be $\Sigma x^{k-1} y$, &c.

If the coefficients contain a new variable in a degree not higher than the power of z, which each coefficient multiplies, then it is easy to see that the result will be not higher than the mnp^{th} degree in this variable.

This, I believe, completes an account of all the principles respecting elimination which have been employed in the text. For fuller details I must refer to those writers who have treated expressly on this subject. I have to state, in conclusion, that Mr. Sylvester had the kindness to communicate to me a very simple rule for the calculation of Aronhold's function, T. The proof of this rule, however, depends on Mr. Sylvester's general researches on subjects connected with elimination, which cannot be explained here without more space than is at my disposal, and more knowledge of the subject than I now possess. Mr. Sylvester's investigations will, I believe, appear in the forthcoming number of the Mathematical Journal, and will, I expect, quite fulfil the anticipations which I have expressed in the note, p. 195.

Mr. Sylvester has also communicated to me a form for the conditions that a cubic should break up into a right line and conic, more convenient than that given, p. 194. The conditions are found by express-

ing that $\frac{dS}{da_1}, \frac{dS}{da_2}$, &c. shall be respectively proportional to $\frac{dT}{da_1}, \frac{dT}{da_2}$, &c. Perhaps it may be possible to obtain a proof of this from the fact remarked in the text, that in this case S and T are proportional to the square and the cube of the same function.

On Imaginary Points and Lines.

Although in the preceding pages we have spoken much of imaginary points and lines, yet we have stated (*Conics*, p. 78) that these are merely analytical conceptions, which we do not attempt to represent geometrically by any real points or lines. M. Poncelet, indeed (who was one of the first to see the importance of giving generality to our geometrical language by the introduction of imaginaries), attempted to make them more intelligible to his readers by giving to these imaginary expressions a real representation. Thus in the equation of a circle $y = \sqrt{(a^2 - x^2)}$, when x is greater than a, and we get a value of the form $y = b\sqrt{-1}$, he represents this by the real point $y = b$, so as to make the equation represent not only the given circle, but also the equilateral hyperbola $y = \sqrt{(x^2 - a^2)}$. This is a mode of representation, however, on which he does not lay much stress, and which, it is obvious, cannot be seriously defended. For it is necessary, to the consistency of our interpretations, that an equation should continue to represent the same curve, however the axes be transformed. And it is plain that in M. Poncelet's method it depends solely on the position of the axes, which, out of an infinity of equilateral hyperbolæ, we are to regard as the companion, also represented by the equation of a circle. A more refined, but, in my opinion, not more successful attempt, to identify imaginary with real points in space, was made by the late Mr. Gregory (Cambridge Mathematical Jour., i. p. 259; Examples, p. 176). I do not know what acceptance Mr. Gregory's method of representation has met with among mathematicians, but still, as it has been introduced in a work to which I have frequently had occasion to refer my readers, I think that I should not be right in silently passing it by, but that I ought to give some account of it, and to state my reasons for not adopting it. And in expressing my dissent I would be understood to do so with all the respect which is due to the memory of one prematurely cut off in so promising a career of mathematical usefulness. Mr. Gregory's views may be stated as follows:—"It is to a certain extent arbitrary what interpretation we give to our algebraical

equations; but the greatest advantage is gained when we adopt the most general methods, and when every algebraical symbol has its appropriate geometrical representation. Thus the inventors of analytic geometry might, if they pleased, have left the sign $-$ uninterpreted, and, confining their attention to the positive values of the variables, have only considered those branches of a curve which lie in the upper right-hand angle between the axes. It was soon seen, however, how much generality might be gained by interpreting the line $-a$ as of equal length, but opposite direction, to the line $+a$; and no curve is now considered as completely traced unless the negative, as well as the positive values of the variables be taken into account. This, however, is merely a matter of convention, and we might, if we pleased, have restricted ourselves to the positive values of the co-ordinates. There remains still, however, the symbol $\sqrt{-1}$, which, in the ordinary systems, is left uninterpreted. Now since -1 denotes the operation of reversing the direction of a line, or of turning the line through 180°, it is natural to consider $(-)^{\frac{1}{2}}$ as that operation which, twice repeated, will give $-$, or the operation of turning the line through 90°. And so in general $a\left(\cos\frac{1}{n}2\pi+\sqrt{-1}\,\sin\frac{1}{n}2\pi\right)$, or $(+)^{\frac{1}{n}}a$, may be considered as denoting a line of the length a inclined at an angle $\frac{2\pi}{n}$ to the line on which $+a$ is measured. When, then, to any value of x there corresponds a value for y of the form just written, this is not to be measured along the axis of y, but along a line inclined to *it* at an angle $\frac{2\pi}{n}$, but still perpendicular to the axis of x, and therefore out of the original plane. This system of interpretation is quite as legitimate an extension as that of the negative values of the variables, and is as necessary to the thorough understanding the course of a curve. When, therefore, the value of either of the co-ordinates ceases to be real, the curve does not cease to exist, but merely leaves the plane of the axes; and we can at all times assign the point in space which corresponds to any value of the co-ordinates. Strictly speaking, then, there are no plane curves, but every curve, part of whose course may be plane, is accompanied by branches in space; and the ordinary methods of interpretation, which take no account of these branches in space, give as imperfect a notion of the course of a curve as a system which should take no account of the negative values of the co-ordinates. By this method also we get a clear conception of conjugate points, which arise when one of the branches in space crosses the plane of the axes."

Now on these views I would remark, that our method of interpretation is not so wholly conventional as Mr. Gregory represents. It is necessary, in the first place, that our interpretations should be consistent; thus, had we commenced by rejecting the negative values of the variables, we should have discovered the incompleteness of our method, by finding that on transformation of co-ordinates our equations had no longer the same geometrical signification. But it is more important to observe, that we do not obtain our first knowledge of geometrical figures from interpreting the equations of Analytic Geometry; but that, on the contrary, our interpretations of analytical equations must be made to coincide with our previous geometrical knowledge. In what I am about to say I shall confine myself to the simplest examples, because many a reader who would not think it incredible that a curve of the third or fourth degree should take excursions out of its plane, will not be disposed to believe in such wanderings if attributed to a right line or a circle. We know what a circle is before we know anything about the equation $x^2 + y^2 = a^2$, and any interpretation of this equation differing either by defect or excess from our previous geometrical conception, must be rejected. We discover that we should be wrong in leaving the sign $-$ uninterpreted, because then the equation $x^2 + y^2 = a^2$ would only represent a fragment of a circle; and we may in the same manner discover that it is objectionable to give a real interpretation to the symbol $\sqrt{-1}$ in the equations of Analytic Geometry, because then $x^2 + y^2 = a^2$ represents not only a circle but an irrelevant curve besides.

In Mr. Gregory's first papers, when he came to an imaginary value, he usually contented himself with saying that the curve left the plane of reference, without much troubling himself to inquire where it went to. This omission was supplied in a Paper by Mr. Walton (Cambridge Mathematical Journal, ii. 103), which I cannot but regard as an able *reductio ad absurdum* of Mr. Gregory's theory. He finds that, according to Mr. Gregory's principles, properly generalized, the equation $x^2 + y^2 = a^2$ represents not only a circle, but also a curve of the fourth degree in space, whose ordinary equations would be written

$$x^2 + y^2 - x^2 \tan^2 2r\pi - (z - x \tan 2r\pi)^2 = a^2 \cos 4m\pi,$$
$$2x(x - y) \tan 2r\pi + 2yz = a^2 \sin 4m\pi,$$

where, however, r is arbitrary; and we are as much at liberty to choose which of an infinity of such curves we are to regard as the companion of the circle, as in M. Poncelet's method we were left to choose between an infinity of equilateral hyperbolæ. Now if it can be shown that our

ordinary geometrical notions of a circle are defective, and that one or all of these curves possess the same properties, and are entitled to be regarded as a portion of the same curve, then Mr. Gregory's mode of interpretation must be hailed as a great discovery. But if these curves differ from a circle in form and properties, then it is an abuse of language to speak of them as branches of the circle, merely because they can be represented by the same equation (and of course what is true of the circle is in like manner true of higher curves); it is to confound two distinct ideas, because they can be expressed by the same symbol; it is, in short, no better than a mathematical pun. And to talk of plane curves having branches out of their plane appears to me calculated to confound all a student's ideas, to make him likely to lose his hold of the principle that a curve of the n^{th} degree is always cut by a right line in n points, and to make him fancy that he has distinct conceptions, which he cannot possibly have, of imaginary points and lines.

The explanation given by Mr. Gregory's method of conjugate points does not show why these points should be always double points; and at any rate the relation of such a point (considered as the limit of an oval), to the curve, seems otherwise sufficiently intelligible. The same theory has been applied by Mr. Gregory or Mr. Walton to the explanation of an impossible branch of a curve having a real asymptote; but this is nothing more than a conjugate point at infinity.

But though I do not admit that any real points can be considered as the representatives of the imaginary points which we are led to in the equations of analytic geometry, I would not be understood to deny that imaginary symbols may be so interpreted as to represent real points and lines. On the contrary, I shall here give a short account of the principles of the symbolical geometry laid down by Mr. Warren in his Tract on the Geometrical Interpretation of Imaginary Quantities, and which to a certain extent coincide with those adopted by Sir William Hamilton in his Quaternion Theory. In this theory all the symbols, +, −, &c. receive new significations, and are to be interpreted with regard to direction as well as to magnitude. Let A, B, C be any three points; let the line a (= AB) denote the operation of passing from the point A to B; let b (= BC), denote the operation of passing from B to C, then the sign + is interpreted to denote that the operations are to be made successively, and we have obviously $AB + BC = AC$, since the result of passing first from A to B, then from B to C, is equivalent to passing at once from A to C. This coincides with the ordinary use of the sign + when the points A, B, C are in one right line. Lines are said

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THE END.